Dynamics in Several Complex Variables: Endomorphisms of Projective Spaces and Polynomial-like Mappings

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Abstract The emphasis of this introductory course is on pluripotential methods in complex dynamics in higher dimension. They are based on the compactness properties of plurisubharmonic (p.s.h.) functions and on the theory of positive closed currents. Applications of these methods are not limited to the dynamical systems that we consider here. Nervertheless, we choose to show their effectiveness and to describe the theory for two large families of maps: the endomorphisms of projective spaces and the polynomial-like mappings. The first section deals with holomorphic endomorphisms of the projective space $\mathbb{P}^k$. We establish the first properties and give several constructions for the Green currents T^p and the equilibrium measure $= T^k$. The emphasis is on quantitative properties and speed of convergence. We then treat equidistribution problems. We show the existence of a proper algebraic set $\mathscr{E}$, totally invariant, i.e. $f^{-1}(\mathscr{E}) = f(\mathscr{E}) = \mathscr{E}$, such that when $a \notin \mathscr{E}$, the probability measures, equidistributed on the fibers $f^{-n}(a)$, converge towards the equilibrium measure , as n goes to infinity. A similar result holds for the restriction of f to invariant subvarieties. We survey the equidistribution problem when points are replaced with varieties of arbitrary dimension, and discuss the equidistribution of periodic points. We then establish ergodic properties of : K-mixing, exponential decay of correlations for various classes of observables, central limit theorem and large deviations theorem. We heavily use the compactness of the space $\mathrm{DSH}(\mathbb{P}^k)$ of differences of quasi-p.s.h. functions. In particular, we show that the measure is moderate, i.e. $\langle , e^{\alpha|\varphi|}\rangle \leq c$, on bounded sets of φ in $\mathrm{DSH}(\mathbb{P}^k)$, for suitable positive constants α, c. Finally, we study the entropy, the Lyapounov exponents and the dimension of . The second section develops the theory of polynomial-like maps, i.e. proper

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holomorphic maps $f : U \to V$ where U, V are open subsets of $\mathbb{C}^k$ with V convex and $U \Subset V$. We introduce the dynamical degrees for such maps and construct the equilibrium measure   of maximal entropy. Then, under a natural assumption on the dynamical degrees, we prove equidistribution properties of points and various statistical properties of the measure   . The assumption is stable under small pertubations on the map. We also study the dimension of   , the Lyapounov exponents and their variation. Our aim is to get a self-contained text that requires only a minimal background. In order to help the reader, an appendix gives the basics on p.s.h. functions, positive closed currents and super-potentials on projective spaces. Some exercises are proposed and an extensive bibliography is given.

Introduction

These notes are based on a series of lectures given by the authors at IHP in 2003, Luminy in 2007, Cetraro in 2008 and Bedlewo 2008. The purpose is to provide an introduction to some developments in dynamics of several complex variables. We have chosen to treat here only two sections of the theory: the dynamics of endomorphisms of the projective space $\mathbb{P}^k$ and the dynamics of polynomial-like mappings in higher dimension. Besides the basic notions and results, we describe the recent developments and the new tools introduced in the theory. These tools are useful in other settings. We tried to give a complete picture of the theory for the above families of dynamical systems. Meromorphic maps on compact Kähler manifolds, in particular polynomial automorphisms of $\mathbb{C}^k$, will be studied in a forthcoming survey.

Let us comment on how complex dynamics fits in the general theory of dynamical systems. The abstract ergodic theory is well-developed with remarkable achievements like the Oseledec-Pesin theory. It is however difficult to show in concrete examples that an invariant measure satisfies exponential decay of correlations for smooth observables or is hyperbolic, i.e. has only non-zero Lyapounov exponents, see e.g. Benedicks-Carleson [BC], Viana [V], L.S. Young [Y, Y1]. One of our goals is to show that holomorphic dynamics in several variables provides remarkable examples of non-uniformly hyperbolic systems where the abstract theory can be applied. Powerful tools from the theory of several complex variables permit to avoid delicate combinatorial estimates. Complex dynamics also require a development of new tools like the calculus on currents and the introduction of new spaces of observables, which are of independent interest.

Complex dynamics in dimension one, i.e. dynamics of rational maps on $\mathbb{P}^1$, is well-developed and has in some sense reached maturity. The main tools there are Montel's theorem on normal families, the Riemann measurable mapping theorem and the theory of quasi-conformal maps, see e.g. Beardon, Carleson-Gamelin [BE, CG]. When dealing with maps in several variables such tools are not available: the Kobayashi hyperbolicity of a manifold and the possibility to apply normal family arguments, are more difficult to check. Holomorphic maps in several variables are not conformal and there is no Riemann measurable mapping theorem.

The theory in higher dimension is developed using mostly pluripotential theory, i.e. the theory of plurisubharmonic (p.s.h. for short) functions and positive closed currents. The Montel's compactness property is replaced with the compactness properties of p.s.h. or quasi-p.s.h. functions. Another crucial tool is the use of good estimates for the dd^c-equation in various settings. One of the main ideas is: in order to study the statistical behavior of orbits of a holomorphic map, we consider its action on some appropriate functional spaces. We then decompose the action into the "harmonic" part and the "non-harmonic" one. This is done solving a dd^c-equation with estimates. The non-harmonic part of the dynamical action may be controled thanks to good estimates for the solutions of a dd^c-equation. The harmonic part can be treated using either Harnack's inequality in the local setting or the linear action of maps on cohomology groups in the case of dynamics on compact Kähler manifolds. This approach has permitted to give a satisfactory theory of the ergodic properties of holomorphic and meromorphic dynamical systems: construction of the measure of maximal entropy, decay of correlations, central limit theorem, large deviations theorem, etc. with respect to that measure.

In order to use the pluripotential methods, we are led to develop the calculus on positive closed currents. Readers not familiar with these theories may start with the appendix at the end of these notes where we have gathered some notions and results on currents and pluripotential theory. A large part in the appendix is classical but there are also some recent results, mostly on new spaces of currents and on the notion of super-potential associated to positive closed currents in higher bidegree. Since we only deal here with projective spaces and open sets in $\mathbb{C}^k$, this is easier and the background is limited.

The main problem in the dynamical study of a map is to understand the behavior of the orbits of points under the action of the map. Simple examples show that in general there is a set (Julia set) where the dynamics is unstable: the orbits may diverge exponentially. Moreover, the geometry of the Julia set is in general very wild. In order to study complex dynamical systems, we follow the classical concepts. We introduce and establish basic properties of some invariants associated to the system, like the topological entropy and the dynamical degrees which are the analogues of volume growth indicators in the real dynamical setting. These invariants give a rough classification of the system. The remarkable fact in complex dynamics is that they can be computed or estimated in many non-trivial situations.

A central question in dynamics is to construct interesting invariant measures, in particular, measures with positive entropy. Metric entropy is an indicator of the complexity of the system with respect to an invariant measure. We focus our study on the measure of maximal entropy. Its support is in some sense the most chaotic part of the system. For the maps we consider here, measures of maximal entropy are constructed using pluripotential methods. For endomorphisms in $\mathbb{P}^k$, they can be obtained as self-intersections of some invariant positive closed $(1,1)$-currents (Green currents). We give estimates on the Hausdorff dimension and on Lyapounov exponents of these measures. The results give the behavior on the most chaotic part. Lyapounov exponents are shown to be strictly positive. This means in some sense that the system is expansive in all directions, despite of the existence of a critical set.

Once, the measure of maximal entropy is constructed, we study its fine dynamical properties. Typical orbits can be observed using test functions. Under the action of the map, each observable provides a sequence of functions that can be seen as dependent random variables. The aim is to show that the dependence is weak and then to establish stochastic properties which are known for independent random variables in probability theory. Mixing, decay of correlations, central limit theorem, large deviations theorems, etc. are proved for the measure of maximal entropy. It is crucial here that the Green currents and the measures of maximal entropy are obtained using an iterative process with estimates; we can then bound the speed of convergence.

Another problem, we consider in these notes, is the equidistribution of periodic points or of preimages of points with respect to the measure of maximal entropy. For endomorphisms of $\mathbb{P}^k$, we also study the equidistribution of varieties with respect to the Green currents. Results in this direction give some informations on the rigidity of the system and also some strong ergodic properties that the Green currents or the measure of maximal entropy satisfy. The results we obtain are in spirit similar to a second main theorem in value distribution theory and should be useful in order to study the arithmetic analogues. We give complete proofs for most results, but we only survey the equidistribution of hypersurfaces and results using super-potentials, in particular, the equidistribution of subvarieties of higher codimension.

The text is organized as follows. In the first section, we study holomorphic endomorphisms of $\mathbb{P}^k$. We introduce several methods in order to construct and to study the Green currents and the Green measure, i.e. equilibrium measure or measure of maximal entropy. These methods were not originally introduced in this setting but here they are simple and very effective. The reader will find a description and the references of the earlier approach in the ten years old survey by the second author [S3]. The second section deals with a very large family of maps: polynomial-like maps. In this case, $f : U \to V$ is proper and defined on an open set U, strictly contained in a convex domain V of $\mathbb{C}^k$. Holomorphic endomorphisms of $\mathbb{P}^k$ can be lifted to a polynomial-like maps on some open set in $\mathbb{C}^{k+1}$. So, we can consider polynomial-like maps as a semi-local version of the endomorphisms studied in the first section. They can appear in the study of meromorphic maps or in the dynamics of transcendental maps. The reader will find in the end of these notes an appendix on the theory of currents and an extensive bibliography. We have given exercises, basically in each section, some of them are not straightforward.

1 Endomorphisms of Projective Spaces

In this section, we give the main results on the dynamics of holomorphic maps on the projective space $\mathbb{P}^k$. Several results are recent and some of them are new even in dimension 1. The reader will find here an introduction to methods that can be developed in other situations, in particular, in the study of meromorphic maps on arbitrary compact Kähler manifolds. The main references for this section are [BD1, BD2, DNS, DS9, DS10, FS1, S3].

1.1 Basic Properties and Examples

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism. Such a map is always induced by a polynomial self-map $F = (F_0, \ldots, F_k)$ on $\mathbb{C}^{k+1}$ such that $F^{-1}(0) = \{0\}$ and the components F_i are homogeneous polynomials of the same degree $d \geq 1$. Given an endomorphism f, the associated map F is unique up to a multiplicative constant and is called *a lift* of f to $\mathbb{C}^{k+1}$. From now on, assume that f is non-invertible, i.e. the *algebraic degree d* is at least 2. Dynamics of an invertible map is simple to study. If $\pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k$ is the natural projection, we have $f \circ \pi = \pi \circ F$. Therefore, dynamics of holomorphic maps on $\mathbb{P}^k$ can be deduced from the polynomial case in $\mathbb{C}^{k+1}$. We will count preimages of points, periodic points, introduce Fatou and Julia sets and give some examples.

It is easy to construct examples of holomorphic maps in $\mathbb{P}^k$. The family of homogeneous polynomial maps F of a given degree d is parametrized by a complex vector space of dimension $N_{k,d} := (k+1)(d+k)!/(d!k!)$. The maps satisfying $F^{-1}(0) = \{0\}$ define a Zariski dense open set. Therefore, the parameter space $\mathcal{H}_d(\mathbb{P}^k)$, of holomorphic endomorphisms of algebraic degree d, is a Zariski dense open set in $\mathbb{P}^{N_{k,d}-1}$, in particular, it is connected.

If $f : \mathbb{C}^k \to \mathbb{C}^k$ is a polynomial map, we can extend f to $\mathbb{P}^k$ but the extension is not always holomorphic. The extension is holomorphic when the dominant homogeneous part f^+ of f, satisfies $(f^+)^{-1}(0) = \{0\}$. Here, if d is the maximal degree in the polynomial expression of f, then f^+ is composed by the monomials of degree d in the components of f. So, it is easy to construct examples using products of one dimensional polynomials or their pertubations.

A general meromorphic map $f : \mathbb{P}^k \to \mathbb{P}^k$ of algebraic degree d is given in homogeneous coordinates by

$$f[z_0 : \cdots : z_k] = [F_0 : \cdots : F_k],$$

where the components F_i are homogeneous polynomials of degree d without common factor, except constants. The map $F := (F_0, \ldots, F_k)$ on $\mathbb{C}^{k+1}$ is still called *a lift* of f. In general, f is not defined on the analytic set $I = \{[z] \in \mathbb{P}^k, F(z) = 0\}$ which is of codimension ≥ 2 since the F_i's have no common factor. This is the *indeterminacy set of f* which is empty when f is holomorphic.

It is easy to check that if f is in $\mathcal{H}_d(\mathbb{P}^k)$ and g is in $\mathcal{H}_{d'}(\mathbb{P}^k)$, the composition $f \circ g$ belongs to $\mathcal{H}_{dd'}(\mathbb{P}^k)$. This is in general false for meromorphic maps: the algebraic degree of the composition is not necessarily equal to the product of the algebraic degrees. It is enough to consider the meromorphic involution of algebraic degree k

$$f[z_0 : \cdots : z_k] := \left[\frac{1}{z_0} : \cdots : \frac{1}{z_k} \right] = \left[\frac{z_0 \cdots z_k}{z_0} : \cdots : \frac{z_0 \cdots z_k}{z_k} \right].$$

The composition $f \circ f$ is the identity map.

We say that f is *dominant* if $f(\mathbb{P}^k \setminus I)$ contains a non-empty open set. The space of dominant meromorphic maps of algebraic degree d, is denoted by $\mathcal{M}_d(\mathbb{P}^k)$. It is also a Zariski dense open set in $\mathbb{P}^{N_{k,d}-1}$. A result by Guelfand, Kapranov and

Zelevinsky shows that $\mathscr{M}_d(\mathbb{P}^k) \setminus \mathscr{H}_d(\mathbb{P}^k)$ is an irreducible algebraic variety [GK]. We will be concerned in this section mostly with holomorphic maps. We show that they are open and their topological degree, i.e. the number of points in a generic fiber, is equal to d^k. We recall here the classical Bézout's theorem which is a central tool for the dynamics in $\mathbb{P}^k$.

Theorem 1.1 (Bézout). *Let $P_1, \ldots, P_k$ be homogeneous polynomials in $\mathbb{C}^{k+1}$ of degrees $d_1, \ldots, d_k$ respectively. Let Z denote the set of common zeros of P_i, in $\mathbb{P}^k$, i.e. the set of points $[z]$ such that $P_i(z) = 0$ for $1 \leq i \leq k$. If Z is discrete, then the number of points in Z, counted with multiplicity, is $d_1 \ldots d_k$.*

The multiplicity of a point a in Z can be defined in several ways. For instance, if U is a small neighbourhood of a and if P_i' are generic homogeneous polynomials of degrees d_i close enough to P_i, then the hypersurfaces $\{P_i' = 0\}$ in $\mathbb{P}^k$ intersect transversally. The number of points of the intersection in U does not depend on the choice of P_i' and is *the multiplicity* of a in Z.

Proposition 1.2. *Let f be an endomorphism of algebraic degree d of $\mathbb{P}^k$. Then for every a in $\mathbb{P}^k$, the fiber $f^{-1}(a)$ contains exactly d^k points, counted with multiplicity. In particular, f is open and defines a ramified covering of degree d^k.*

Proof. For the multiplicity of f and the notion of ramified covering, we refer to Appendix A.1. Let $f = [F_0 : \cdots : F_k]$ be an expression of f in homogeneous coordinates. Consider a point $a = [a_0 : \cdots : a_k]$ in $\mathbb{P}^k$. Without loss of generality, we can assume $a_0 = 1$, hence $a = [1 : a_1 : \cdots : a_k]$. The points in $f^{-1}(a)$ are the common zeros, in $\mathbb{P}^k$, of the polynomials $F_i - a_i F_0$ for $i = 1, \ldots, k$.

We have to check that the common zero set is discrete, then Bézout's theorem asserts that the cardinality of this set is equal to the product of the degrees of $F_i - a_i F_0$, i.e. to d^k. If the set were not discrete, then the common zero set of $F_i - a_i F_0$ in $\mathbb{C}^{k+1}$ is analytic of dimension ≥ 2. This implies that the set of common zeros of the F_i's, $0 \leq i \leq k$, in $\mathbb{C}^{k+1}$ is of positive dimension. This is impossible when f is holomorphic. So, f is a ramified covering of degree d^k. In particular, it is open.

Note that when f is a map in $\mathscr{M}_d(\mathbb{P}^k) \setminus \mathscr{H}_d(\mathbb{P}^k)$ with indeterminacy set I, we can prove that the generic fibers of $f : \mathbb{P}^k \setminus I \to \mathbb{P}^k$ contains at most $d^k - 1$ points. Indeed, for every a, the hypersurfaces $\{F_i - a_i F_0 = 0\}$ in $\mathbb{P}^k$ contain I. $\qquad\square$

Periodic points of order n, i.e. points which satisfy $f^n(z) = z$, play an important role in dynamics. Here, $f^n := f \circ \cdots \circ f$, n times, is *the iterate of order n of f*. Periodic points of order n of f are fixed points of f^n which is an endomorphism of algebraic degree d^n. In the present case, their counting is simple. We have the following result.

Proposition 1.3. *Let f be an endomorphism of algebraic degree $d \geq 2$ in $\mathbb{P}^k$. Then the number of fixed points of f, counted with multiplicity, is equal to $(d^{k+1} - 1)/(d - 1)$. In particular, the number of periodic points of order n of f is $d^{kn} + o(d^{kn})$.*

Proof. There are several methods to count the periodic points. In $\mathbb{P}^{k+1}$, with homogeneous coordinates $[z:t] = [z_0 : \cdots : z_k : t]$, we consider the system of equations $F_i(z) - t^{d-1}z_i = 0$. The set is discrete since it is analytic and does not intersect the hyperplane $\{t = 0\}$. So, we can count the solutions of the above system using Bézout's theorem and we find d^{k+1} points counting with multiplicity. Each point $[z:t]$ in this set, except $[0:\cdots:0:1]$, corresponds to a fixed point $[z]$ of f. The correspondence is $d-1$ to 1. Indeed, if we multiply t by a $(d-1)$-th root of unity, we get the same fixed point. Hence, the number of fixed points of f counted with multiplicity is $(d^{k+1} - 1)/(d-1)$.

The number of fixed points of f is also the number of points in the intersection of the graph of f with the diagonal of $\mathbb{P}^k \times \mathbb{P}^k$. So, we can count these points using the cohomology classes associated to the above analytic sets, i.e. using the Lefschetz fixed point formula, see [GH]. We can also observe that this number depends continuously on f. So, it is constant for f in $\mathscr{H}_d(\mathbb{P}^k)$ which is connected. We obtain the result by counting the fixed points of an example, e.g. for $f[z] = [z_0^d : \cdots : z_k^d]$. □

Note that the periodic points of period n are isolated. If p is such a point, a theorem due to Shub-Sullivan [KH, p.323] implies that the multiplicity at p of the equation $f^{mn}(p) = p$ is bounded independently on m. The result holds for $\mathscr{C}^1$ maps. We deduce from the above result that f admits infinitely many distinct periodic points.

The set of fixed points of a meromorphic map could be empty or infinite. One checks easily that the map $(z_1, z_2) \mapsto (z_1^2, z_2)$ in $\mathbb{C}^2$ admits $\{z_1 = 0\}$ as a curve of fixed points.

Example 1.4. Consider the following map:

$$f(z_1, z_2) := (z_1 + 1, P(z_1, z_2)),$$

where P is a homogeneous polynomial of degree $d \geq 2$ such that $P(0,1) = 0$. It is clear that f has no periodic point in $\mathbb{C}^2$. The meromorphic extension of f is given in homogeneous coordinates $[z_0 : z_1 : z_2]$ by

$$f[z] = [z_0^d : z_0^{d-1}z_1 + z_0^d : P(z_1, z_2)].$$

Here, $\mathbb{C}^2$ is identified to the open set $\{z_0 = 1\}$ of $\mathbb{P}^2$. The indeterminacy set I of f is defined by $z_0 = P(z_1, z_2) = 0$ and is contained in the line at infinity $L_\infty := \{z_0 = 0\}$. We have $f(L_\infty \setminus I) = [0:0:1]$ which is an indeterminacy point. So, $f : \mathbb{P}^2 \setminus I \to \mathbb{P}^2$ has no periodic point.

Example 1.5. Consider the holomorphic map f on $\mathbb{P}^2$ given by

$$f[z] := [z_0^d + P(z_1, z_2) : z_2^d + \lambda z_0^{d-1}z_1 : z_1^d],$$

with P homogeneous of degree $d \geq 2$. Let $p := [1:0:0]$, then $f^{-1}(p) = p$. Such a point is called *totally invariant*. In general, p is not necessarily an attractive point. Indeed, the eigenvalues of the differential of f at p are 0 and λ. When $|\lambda| > 1$, there is an expansive direction for f in a neighbourhood of p. In dimension one, totally invariant points are always attractive.

For a holomorphic map f on $\mathbb{P}^k$, a point a in $\mathbb{P}^k$ is *critical* if f is not injective in a neighbourhood of a or equivalently the multiplicity of f at a in the fiber $f^{-1}(f(a))$ is strictly larger than 1, see Theorem A.3. We say that a is a critical point of *multiplicity m* if the multiplicity of f at a in the fiber $f^{-1}(f(a))$ is equal to $m+1$.

Proposition 1.6. *Let f be a holomorphic endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$. Then, the critical set of f is an algebraic hypersurface of degree $(k+1)(d-1)$ counted with multiplicity.*

Proof. If F is a lift of f to $\mathbb{C}^{k+1}$, the Jacobian $\mathrm{Jac}(F)$ is a homogeneous polynomial of degree $(k+1)(d-1)$. The zero set of $\mathrm{Jac}(F)$ in $\mathbb{P}^k$ is exactly the critical set of f. The result follows. $\square$

Let $\mathscr{C}$ denote the critical set of f. The orbit $\mathscr{C}, f(\mathscr{C}), f^2(\mathscr{C}), \ldots$ is either a hypersurface or a countable union of hypersurfaces. We say that f is *postcritically finite* if this orbit is a hypersurface, i.e. has only finitely many irreducible components. Besides very simple examples, postcritically finite maps are difficult to construct, because the image of a variety is often a variety of larger degree; so we have to increase the multiplicity in order to get only finitely many irreducible components. We give few examples of postcritically finite maps, see [FS, FS7].

Example 1.7. We can check that for $d \geq 2$ and $(1 - 2\lambda)^d = 1$

$$f[z_0 : \cdots : z_k] := [z_0^d : \lambda(z_0 - 2z_1)^d : \cdots : \lambda(z_0 - 2z_k)^d]$$

is postcritically finite. For some parameters $\alpha \in \mathbb{C}$ and $0 \leq l \leq d$, the map

$$f_\alpha[z] := [z_0^d : z_1^d : z_2^d + \alpha z_1^{d-l} z_2^l]$$

is also postcritically finite. In particular, for $f_0[z] = [z_0^d : z_1^d : z_2^d]$, the associated critical set is equal to $\{z_0 z_1 z_2 = 0\}$ which is invariant under f_0. So, f_0 is postcritically finite.

Arguing as above, using Bézout's theorem, we can prove that if Y is an analytic set of pure codimension p in $\mathbb{P}^k$ then $f^{-1}(Y)$ is an analytic set of pure codimension p. Its degree, counting with multiplicity, is equal to $d^p \deg(Y)$. Recall that the degree $\deg(Y)$ of Y is the number of points in the intersection of Y with a generic projective subspace of dimension p. We deduce that the pull-back operator f^* on the Hodge cohomology group $H^{p,p}(\mathbb{P}^k, \mathbb{C})$ is simply a multiplication by d^p. Since f is a ramified covering of degree d^k, $f_* \circ f^*$ is the multiplication by d^k. Therefore, the push-forward operator f_* acting on $H^{p,p}(\mathbb{P}^k, \mathbb{C})$ is the multiplication by d^{k-p}. In particular, the image $f(Y)$ of Y by f is an analytic set of pure codimension p and of degree $d^{k-p} \deg(Y)$, counted with multiplicity.

We now introduce the Fatou and Julia sets associated to an endomorphism. The following definition is analogous to the one variable case.

Definition 1.8. *The Fatou set of f is the largest open set $\mathscr{F}$ in $\mathbb{P}^k$ where the sequence of iterates $(f^n)_{n \geq 1}$ is locally equicontinuous. The complement $\mathscr{J}$ of $\mathscr{F}$ is called the Julia set of f.*

Fatou and Julia sets are totally invariant by $\mathscr{F}$, that is, $f^{-1}(\mathscr{F}) = f(\mathscr{F}) = \mathscr{F}$ and the same property holds for $\mathscr{J}$. Julia and Fatou sets associated to f^n are also equal to $\mathscr{J}$ and $\mathscr{F}$. We see here that the space $\mathbb{P}^k$ is divided into two parts: on $\mathscr{F}$ the dynamics is stable and tame while the dynamics on $\mathscr{J}$ is a priori chaotic. If x is a point in $\mathscr{F}$ and y is close enough to x, the orbit of y is close to the orbit of x when the time n goes to infinity. On the Julia set, this property is not true. Attractive fixed points and their basins are examples of points in the Fatou set. Siegel domains, i.e. invariant domains on which f is conjugated to a rotation, are also in the Fatou set. Repelling periodic points are always in the Julia set. Another important notion in dynamics is the non-wandering set.

Definition 1.9. A point a in $\mathbb{P}^k$ is *non-wandering* with respect to f if for every neighbourhood U of a, there is an $n \geq 1$ such that $f^n(U) \cap U \neq \varnothing$.

The study of the Julia and Fatou sets is a fundamental problem in dynamics. It is quite well-understood in the one variable case where the Riemann measurable theorem is a basic tool. The help of computers is also important there. In higher dimension, Riemann measurable theorem is not valid and the use of computers is more delicate. The most important tool in higher dimension is pluripotential theory.

For instance, Fatou and Julia sets for a general map are far from being understood. Many fundamental questions are still open. We do not know if wandering Fatou components exist in higher dimension. In dimension one, a theorem due to Sullivan [SU] says that such a domain does not exist. The classification of Fatou components is not known, see [FS8] for a partial answer in dimension 2 and [FS, S3, U] for the case of postcritically finite maps. The reader will find in the survey [S3] some results on local dynamics near a fixed point, related to the Fatou-Julia theory. We now give few examples.

The following construction is due to Ueda [U]. It is useful in order to obtain interesting examples, in particular, to show that some properties hold for generic maps. The strategy is to check that the set of maps satisfying these properties is a Zariski open set in the space of parameters and then to produce an example using Ueda's construction.

Example 1.10. Let $h : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d \geq 2$. Consider the multi-projective space $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$, k times. The permutations of coordinates define a finite group Γ acting on this space and the quotient of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ by Γ is equal to $\mathbb{P}^k$. Let $\Pi : \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \to \mathbb{P}^k$ denote the canonical projection. Let $\widetilde{f}$ be the endomorphism of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ defined by $\widetilde{f}(z_1, \ldots, z_k) := (h(z_1), \ldots, h(z_k))$. If σ is a permutation of coordinates $(z_1, \ldots, z_k)$, then $\sigma \circ \widetilde{f} = \widetilde{f} \circ \sigma$. It is not difficult to deduce that there is an endomorphism f on $\mathbb{P}^k$ of algebraic degree d semi-conjugated to $\widetilde{f}$, that is, $f \circ \Pi = \Pi \circ \widetilde{f}$. One can deduce dynamical properties of f from properties of h. For example, if h is chaotic, i.e. has a dense orbit, then f is also chaotic. The first chaotic maps on $\mathbb{P}^1$ were constructed by Lattès. Ueda's construction gives Lattès maps in higher dimension. A *Lattès map* f on $\mathbb{P}^k$ is a map semi-conjugated to an affine map on a torus. More precisely, there is an open holomorphic map $\Psi : \mathbb{T} \to \mathbb{P}^k$ from a k-dimensional torus $\mathbb{T}$ onto $\mathbb{P}^k$ and an affine map $A : \mathbb{T} \to \mathbb{T}$ such that $f \circ \Psi = \Psi \circ A$. We refer to [BR, BL, D4, DS0, MIL] for a discussion of Lattès maps.

The following map is the simplest in our context. Its iterates can be explicitly computed. The reader may use this map and its pertubations as basic examples in order to get a picture on the objects we will introduce latter.

Example 1.11. Let $f : \mathbb{C}^k \to \mathbb{C}^k$ be the polynomial map defined by

$$f(z_1, \ldots, z_k) := (z_1^d, \ldots, z_k^d), \quad d \geq 2.$$

We can extend f holomorphically to $\mathbb{P}^k$. Let $[z_0 : \cdots : z_k]$ denote the homogeneous coordinates on $\mathbb{P}^k$ such that $\mathbb{C}^k$ is identified to the chart $\{z_0 \neq 0\}$. Then, the extension of f to $\mathbb{P}^k$ is

$$f[z_0 : \cdots : z_k] = [z_0^d : \cdots : z_k^d].$$

The Fatou set is the union of the basins of the $k + 1$ attractive fixed points $[0 : \cdots : 0 : 1 : 0 : \cdots : 0]$. These components are defined by

$$\mathscr{F}_i := \left\{ z \in \mathbb{P}^k, \quad |z_j| < |z_i| \quad \text{for every } j \neq i \right\}.$$

The Julia set of f is the union of the following sets $\mathscr{J}_{ij}$ with $0 \leq i < j \leq k$, where

$$\mathscr{J}_{ij} := \left\{ z \in \mathbb{P}^k, \quad |z_i| = |z_j| \quad \text{and} \quad |z_l| \leq |z_i| \quad \text{for every } l \right\}.$$

We have $f^n(z) = (z_1^{d^n}, \ldots, z_k^{d^n})$ for $n \geq 1$.

Exercise 1.12. Let $h : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map. Discuss Fatou components for the associated map f defined in Example 1.10. Prove in particular that there exist Fatou components which are bi-holomorphic to a disc cross an annulus. Describe the set of non-wandering points of f.

Exercise 1.13. Let a be a fixed point of f. Show that the eigenvalues of the differential Df of f at a do not depend on the local coordinates. Assume that a is in the Fatou set. Show that these eigenvalues are of modulus ≤ 1. If all the eigenvalues are of modulus 1, show that $Df(a)$ is diagonalizable.

Exercise 1.14. Let f be a Lattès map associated to an affine map A as in Example 1.10. Show that f is postcritically finite. Show that $d^{-1/2}DA$ is an unitary matrix where DA is the differential of A. Deduce that the orbit of a is dense in $\mathbb{P}^k$ for almost every a in $\mathbb{P}^k$. Show that the periodic points of f are dense in $\mathbb{P}^k$.

Exercise 1.15. Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a dominant meromorphic map. Let I be the indeterminacy set of f, defined as above. Show that f cannot be extended to a holomorphic map on any open set which intersects I.

1.2 Green Currents and Julia Sets

Let f be an endomorphism of algebraic degree $d \geq 2$ as above. In this paragraph, we give the first construction of canonical invariant currents T^p associated to f (Green

currents). The construction is now classical and is used in most of the references, see [FS1, FS3, HJ, S3]. We will show that the support of the Green $(1,1)$-current is exactly the Julia set of f [FS1]. In some examples, Green currents describe the distribution of stable varieties but in general their geometric structure is not yet well-understood. We will see later that $\quad := T^k$ is the invariant measure of maximal entropy.

Theorem 1.16. *Let S be a positive closed $(1,1)$-current of mass 1 on $\mathbb{P}^k$. Assume that S has bounded local potentials. Then $d^{-n}(f^n)^*(S)$ converge weakly to a positive closed $(1,1)$-current T of mass 1. This current has continuous local potentials and does not depend on S. Moreover, it is totally invariant: $f^*(T) = dT$ and $f_*(T) = d^{k-1}T$. We also have for a smooth $(k-1,k-1)$-form Φ*

$$\left|\langle d^{-n}(f^n)^*(S) - T, \Phi \rangle\right| \leq cd^{-n}\|\Phi\|_{\mathrm{DSH}},$$

where $c > 0$ is a constant independent of Φ and of n.

Proof. We refer to Appendix for the basic properties of quasi-p.s.h. functions, positive closed currents and DSH currents. Since S has mass 1, it is cohomologous to ω_{FS}. Therefore, we can write $S = \omega_{\mathrm{FS}} + dd^c u$ where u is a quasi-p.s.h. function. By hypothesis, this function is bounded. The current $d^{-1}f^*(\omega_{\mathrm{FS}})$ is smooth and of mass 1 since $f^* : H^{1,1}(\mathbb{P}^k, \mathbb{C}) \to H^{1,1}(\mathbb{P}^k, \mathbb{C})$ is the multiplication by d and the mass of a positive closed current can be computed cohomologically. So, we can also write $d^{-1}f^*(\omega_{\mathrm{FS}}) = \omega_{\mathrm{FS}} + dd^c v$ where v is a quasi-p.s.h. function. Here, v is smooth since ω_{FS} and $f^*(\omega_{\mathrm{FS}})$ are smooth. We have

$$d^{-1}f^*(S) = d^{-1}f^*(\omega_{\mathrm{FS}}) + dd^c\left(d^{-1}u \circ f\right)$$
$$= \omega_{\mathrm{FS}} + dd^c v + dd^c\left(d^{-1}u \circ f\right).$$

By induction, we obtain

$$d^{-n}(f^n)^*(S) = \omega_{\mathrm{FS}} + dd^c\left(v + \cdots + d^{-n+1}v \circ f^{n-1}\right) + dd^c\left(d^{-n}u \circ f^n\right).$$

Observe that, since v is smooth, the sequence of smooth functions $v + \cdots + d^{-n+1}v \circ f^{n-1}$ converges uniformly to a continuous function g. Since u is bounded, the functions $d^{-n}u \circ f^n$ tend to 0. It follows that $d^{-n}(f^n)^*(S)$ converge weakly to a current T which satisfies

$$T = \omega_{\mathrm{FS}} + dd^c g.$$

From the definition, we also have

$$d^{-1}g \circ f + v = g.$$

Clearly, this current does not depend on S since g does not depend on S. Moreover, the currents $d^{-n}(f^n)^*(S)$ are positive closed of mass 1. So, T is also a positive closed current of mass 1. We deduce that g is quasi-p.s.h. since it is continuous and satisfies $dd^c g \geq -\omega_{\mathrm{FS}}$.

Applying the above computation to T instead of S, we obtain that

$$d^{-1}f^*(T) = \omega_{\text{FS}} + dd^c v + dd^c(d^{-1}\text{g} \circ f) = \omega_{\text{FS}} + dd^c\text{g}.$$

Hence, $f^*(T) = dT$. On smooth forms $f_* \circ f^*$ is equal to d^k times the identity; this holds by continuity for positive closed currents. Therefore,

$$f_*(T) = f_*(f^*(d^{-1}T)) = d^{k-1}T.$$

It remains to prove the estimate in the theorem. Recall that we can write $dd^c\Phi = R^+ - R^-$ where $R^{\pm}$ are positive measures such that $\|R^{\pm}\| \leq \|\Phi\|_{\text{DSH}}$. We have

$$\begin{aligned}
\left|\langle d^{-n}(f^n)^*(S) - T, \Phi \rangle\right| &= \left|\langle dd^c(v + \cdots + d^{-n+1}v \circ f^{n-1} + d^{-n}u \circ f^n - \text{g}), \Phi \rangle\right| \\
&= \left|\langle v + \cdots + d^{-n+1}v \circ f^{n-1} + d^{-n}u \circ f^n - \text{g}, dd^c\Phi \rangle\right| \\
&= \left|\langle d^{-n}u \circ f^n - \sum_{i \geq n} d^{-i}v \circ f^i, R^+ - R^- \rangle\right|.
\end{aligned}$$

Since u and v are bounded, the mass estimate for $R^{\pm}$ implies that the last integral is $\lesssim d^{-n}\|\Phi\|_{\text{DSH}}$. The result follows. $\square$

Theorem 1.16 gives a convergence result for S quite diffuse (with bounded potentials). It is like the first main theorem in value distribution theory. The question that we will address is the convergence for singular S, e.g. hypersurfaces.

Definition 1.17. We call T *the Green* $(1,1)$*-current* and g *the Green function* of f. The power $T^p := T \wedge \ldots \wedge T$, p factors, is *the Green* (p,p)*-current* of f, and its support $\mathscr{J}_p$ is called *the Julia set of order p*.

Note that the Green function is defined up to an additive constant and since T has a continuous quasi-potential, T^p is well-defined. Green currents are totally invariant: we have $f^*(T^p) = d^p T^p$ and $f_*(T^p) = d^{k-p}T^p$. The Green (k,k)-current $:= T^k$ is also called *the Green measure, the equilibrium measure or the measure of maximal entropy*. We will give in the next paragraphs results which justify the terminologies. The iterates f^n, $n \geq 1$, have the same Green currents and Green function. We have the following result.

Proposition 1.18. *The local potentials of the Green current T are γ-Hölder continuous for every γ such that $0 < \gamma < \min(1, \log d / \log d_\infty)$, where $d_\infty := \lim \|Df^n\|_\infty^{1/n}$. In particular, the Hausdorff dimension of T^p is strictly larger than $2(k-p)$ and T^p has no mass on pluripolar sets and on proper analytic sets of $\mathbb{P}^k$.*

Since $Df^{n+m}(x) = Df^m(f^n(x)) \circ Df^n(x)$, it is not difficult to check that the sequence $\|Df^n\|_\infty^{1/n}$ is decreasing. So, $d_\infty = \inf\|Df^n\|_\infty^{1/n}$. The last assertion of the proposition is deduced from Corollary A.32 and Proposition A.33 in Appendix. The first assertion is equivalent to the Hölder continuity of the Green function g, it was obtained by Sibony [SI] for one variable polynomials and by Briend [BJ] and Kosek [KO] in higher dimension.

The following lemma, due to Dinh-Sibony [DS4, DS10], implies the above proposition and can be applied in a more general setting. Here, we apply it to $\Lambda := f^m$ with m large enough and to the above smooth function v. We choose $\alpha := 1, A := \|Df^m\|_\infty$ and d is replaced with d^m.

Lemma 1.19. *Let K be a metric space with finite diameter and $\Lambda : K \to K$ be a Lipschitz map: $\|\Lambda(a) - \Lambda(b)\| \leq A\|a - b\|$ with $A > 0$. Here, $\|a - b\|$ denotes the distance between two points a, b in K. Let v be an α-Hölder continuous function on K with $0 < \alpha \leq 1$. Then, $\sum_{n \geq 0} d^{-n} v \circ \Lambda^n$ converges pointwise to a function which is β-Hölder continuous on K for every β such that $0 < \beta < \alpha$ and $\beta \leq \log d / \log A$.*

Proof. By hypothesis, there is a constant $A' > 0$ such that $|v(a) - v(b)| \leq A'\|a - b\|^\alpha$. Define $A'' := \|v\|_\infty$. Since K has finite diameter, A'' is finite and we only have to consider the case where $\|a - b\| \ll 1$. If N is an integer, we have

$$
\left| \sum_{n \geq 0} d^{-n} v \circ \Lambda^n(a) - \sum_{n \geq 0} d^{-n} v \circ \Lambda^n(b) \right|
$$

$$
\leq \sum_{0 \leq n \leq N} d^{-n} |v \circ \Lambda^n(a) - v \circ \Lambda^n(b)| + \sum_{n > N} d^{-n} |v \circ \Lambda^n(a) - v \circ \Lambda^n(b)|
$$

$$
\leq A' \sum_{0 \leq n \leq N} d^{-n} \|\Lambda^n(a) - \Lambda^n(b)\|^\alpha + 2A'' \sum_{n > N} d^{-n}
$$

$$
\lesssim \|a - b\|^\alpha \sum_{0 \leq n \leq N} d^{-n} A^{n\alpha} + d^{-N}.
$$

If $A^\alpha \leq d$, the last sum is of order at most equal to $N\|a - b\|^\alpha + d^{-N}$. For a given $0 < \beta < \alpha$, choose $N \simeq -\beta \log \|a - b\| / \log d$. So, the last expression is $\lesssim \|a - b\|^\beta$. In this case, the function is β-Hölder continuous for every $0 < \beta < \alpha$. When $A^\alpha > d$, the sum is $\lesssim d^{-N} A^{N\alpha} \|a - b\|^\alpha + d^{-N}$. For $N \simeq -\log \|a - b\| / \log A$, the last expression is $\lesssim \|a - b\|^\beta$ with $\beta := \log d / \log A$. Therefore, the function is β-Hölder continuous. $\square$

Remark 1.20. Lemma 1.19 still holds for K with infinite diameter if v is Hölder continuous and bounded. We can also replace the distance on K with any positive symmetric function on $K \times K$ which vanishes on the diagonal. Consider a family (f_s) of endomorphisms of $\mathbb{P}^k$ depending holomorphically on s in a space of parameters Σ. In the above construction of the Green current, we can locally on Σ, choose $v_s(z)$ smooth such that $dd^c_{s,z} v_s(z) \geq -\omega_{FS}(z)$. Lemma 1.19 implies that the Green function $g_s(z)$ of f_s is locally Hölder continuous on (s, z) in $\Sigma \times \mathbb{P}^k$. Then, $\omega_{FS}(z) + dd^c_{s,z} g_s(z)$ is a positive closed $(1,1)$-current on $\Sigma \times \mathbb{P}^k$. Its slices by $\{s\} \times \mathbb{P}^k$ are the Green currents T_s of f_s.

We want to use the properties of the Green currents in order to establish some properties of the Fatou and Julia sets. We will show that the Julia set coincides with the Julia set of order 1. We recall the notion of Kobayashi hyperbolicity on a complex manifold M. Let p be a point in M and ξ a tangent vector of M at p. Consider the holomorphic maps $\tau : \Delta \to M$ on the unit disc Δ in $\mathbb{C}$ such that

$\tau(0) = p$ and $D\tau(0) = c\xi$ where $D\tau$ is the differential of τ and c is a constant. *The Kobayashi-Royden pseudo-metric* is defined by

$$K_M(p,\xi) := \inf_\tau |c|^{-1}.$$

It measures the size of a disc that one can send in M. In particular, if M contains an image of $\mathbb{C}$ passing through p in the direction ξ, we have $K_M(p,\xi) = 0$.

Kobayashi-Royden pseudo-metric is contracting for holomorphic maps: if $\Psi : N \to M$ is a holomorphic map between complex manifolds, we have

$$K_M(\Psi(p), D\Psi(p) \cdot \xi) \leq K_N(p,\xi).$$

The Kobayashi-Royden pseudo-metric on Δ coincides with the Poincaré metric. A complex manifold M is *Kobayashi hyperbolic* if K_M is a metric [K]. In which case, holomorphic self-maps of M, form a locally equicontinuous family of maps. We have the following result where the norm of ξ is with respect to a smooth metric on X.

Proposition 1.21. *Let M be a relatively compact open set of a compact complex manifold X. Assume that there is a bounded function ρ on M which is strictly p.s.h., i.e. $dd^c\rho \geq \omega$ on M for some positive Hermitian form ω on X. Then M is Kobayashi hyperbolic and hyperbolically embedded in X. More precisely, there is a constant $\lambda > 0$ such that $K_M(p,\xi) \geq \lambda \|\xi\|$ for every $p \in M$ and every tangent vector ξ of M at p.*

Proof. If not, we can find holomorphic discs $\tau_n : \Delta \to M$ such that $\|D\tau_n(0)\| \geq n$ for $n \geq 1$. So, this family is not equicontinuous. A lemma due to Brody [K] says that, after reparametrization, there is a subsequence converging to an image of $\mathbb{C}$ in $\overline{M}$. More precisely, up to extracting a subsequence, there are holomorphic maps $\Psi_n : \Delta_n \to \Delta$ on discs Δ_n centered at 0, of radius n, such that $\tau_n \circ \Psi_n$ converge locally uniformly to a non-constant map $\tau_\infty : \mathbb{C} \to \overline{M}$. Since ρ is bounded, up to extracting a subsequence, the subharmonic functions $\rho_n := \rho \circ \tau_n \circ \Psi_n$ converge in $L^1_{loc}(\mathbb{C})$ to some subharmonic function ρ_∞. Since the function ρ_∞ is bounded, it should be constant.

For simplicity, we use here the metric on X induced by ω. Let L, K be arbitrary compact subsets of $\mathbb{C}$ such that $L \Subset K$. For n large enough, the area of $\tau_n(\Psi_n(L))$ counted with multiplicity, satisfies

$$\text{area}(\tau_n(\Psi_n(L))) = \int_L (\tau_n \circ \Psi_n)^*(\omega) \leq \int_L dd^c\rho_n.$$

We deduce that

$$\text{area}(\tau_\infty(L)) = \lim_{n\to\infty} \text{area}(\tau_n(\Psi_n(L))) \leq \int_K dd^c\rho_\infty = 0.$$

This is a contradiction. $\square$

The following result was obtained by Fornæss-Sibony in [FS3, FS2] and by Ueda for the assertion on the Kobayashi hyperbolicity of the Fatou set [U2].

Theorem 1.22. *Let f be an endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$. Then, the Julia set of order 1 of f, i.e. the support $\mathscr{J}_1$ of the Green $(1,1)$-current T, coincides with the Julia set $\mathscr{J}$. The Fatou set $\mathscr{F}$ is Kobayashi hyperbolic and hyperbolically embedded in $\mathbb{P}^k$. Moreover, for $p \leq k/2$, the Julia set of order p of f is connected.*

Proof. The sequence (f^n) is equicontinuous on the Fatou set $\mathscr{F}$ and f^n are holomorphic, hence the differential Df^n are locally uniformly bounded on $\mathscr{F}$. Therefore, $(f^n)^*(\omega_{\mathrm{FS}})$ are locally uniformly bounded on $\mathscr{F}$. We deduce that $d^{-n}(f^n)^*(\omega_{\mathrm{FS}})$ converge to 0 on $\mathscr{F}$. Hence, T is supported on the Julia set $\mathscr{J}$.

Let $\mathscr{F}'$ denote the complement of the support of T in $\mathbb{P}^k$. Observe that $\mathscr{F}'$ is invariant under f^n and that $-g$ is a smooth function which is strictly p.s.h. on $\mathscr{F}'$. Therefore, by Proposition 1.21, $\mathscr{F}'$ is Kobayashi hyperbolic and hyperbolically embedded in $\mathbb{P}^k$. Therefore, the maps f^n, which are self-maps of $\mathscr{F}'$, are equicontinuous with respect to the Kobayashi-Royden metric. On the other hand, the fact that $\mathscr{F}'$ is hyperbolically embedded implies that the Kobayashi-Royden metric is bounded from below by a constant times the Fubini-Study metric. It follows that (f^n) is locally equicontinuous on $\mathscr{F}'$ with respect to the Fubini-Study metric. We conclude that $\mathscr{F}' \subset \mathscr{F}$, hence $\mathscr{F} = \mathscr{F}'$ and $\mathscr{J} = \mathrm{supp}(T) = \mathscr{J}_1$.

In order to show that $\mathscr{J}_p$ are connected, it is enough to prove that if S is a positive closed current of bidegree (p,p) with $p \leq k/2$ then the support of S is connected. Assume that the support of S is not connected, then we can write $S = S_1 + S_2$ with S_1 and S_2 non-zero, positive closed with disjoint supports. Using a convolution on the automorphism group of $\mathbb{P}^k$, we can construct smooth positive closed (p,p)-forms S_1', S_2' with disjoint supports. So, we have $S_1' \wedge S_2' = 0$. This contradicts that the cup-product of the classes $[S_1']$ and $[S_2']$ is non-zero in $H^{2p,2p}(\mathbb{P}^k, \mathbb{R}) \simeq \mathbb{R}$: we have $[S_1'] = \|S_1'\|[\omega_{\mathrm{FS}}^p]$, $[S_2'] = \|S_2'\|[\omega_{\mathrm{FS}}^p]$ and $[S_1'] \smile [S_2'] = \|S_1'\|\|S_2'\|[\omega_{\mathrm{FS}}^{2p}]$, a contradiction. Therefore, the support of S is connected. $\square$

Example 1.23. Let f be a polynomial map of algebraic degree $d \geq 2$ on $\mathbb{C}^k$ which extends holomorphically to $\mathbb{P}^k$. If B is a ball large enough centered at 0, then $f^{-1}(B) \Subset B$. Define $G_n := d^{-n}\log^+\|f^n\|$, where $\log^+ := \max(\log, 0)$. As in Theorem 1.16, we can show that G_n converge uniformly to a continuous p.s.h. function G such that $G \circ f = dG$. On $\mathbb{C}^k$, the Green current T of f is equal to $dd^c G$ and $T^p = (dd^c G)^p$. The Green measure is equal to $(dd^c G)^k$. If $\mathscr{K}$ denotes the set of points in $\mathbb{C}^k$ with bounded orbit, then   is supported on $\mathscr{K}$. Indeed, outside $\mathscr{K}$ we have $G = \lim d^{-n}\log\|f^n\|$ and the convergence is locally uniform. It follows that $(dd^c G)^k = \lim d^{-kn}(dd^c \log\|f^n\|)^k$ on $\mathbb{C}^k \setminus \mathscr{K}$. One easily check that $(dd^c \log\|f^n\|)^k = 0$ out of $f^{-n}(0)$. Therefore, $(dd^c G)^k = 0$ on $\mathbb{C}^k \setminus \mathscr{K}$. The set $\mathscr{K}$ is called *the filled Julia set*. We can show that $\mathscr{K}$ is the zero set of G. In particular, if $f(z) = (z_1^d, \ldots, z_k^d)$, then $G(z) = \sup_i \log^+ |z_i|$. One can check that the support of T^p is foliated (except for a set of zero measure with respect to the trace of T^p) by stable manifolds of dimension $k - p$ and that $= T^k$ is the Lebesgue measure on the torus $\{|z_i| = 1, \ i = 1, \ldots, k\}$.

Example 1.24. We consider Example 1.10. Let ν be the Green measure of h on $\mathbb{P}^1$, i.e. $\nu = \lim d^{-n}(h^n)^*(\omega_{\mathrm{FS}})$. Here, ω_{FS} denotes also the Fubini-Study form on $\mathbb{P}^1$.

Let π_i denote the projections of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ on the factors. Then, the Green current of f is equal to

$$T = \frac{1}{k!} \pi_* \big(\pi_1^*(v) + \cdots + \pi_k^*(v) \big),$$

as can be easily checked.

Example 1.25. The following family of maps on $\mathbb{P}^2$ was studied in [FS6]:

$$f[z_0 : z_1 : z_2] := [z_0^d + \lambda z_0 z_1^{d-1}, v(z_1 - 2z_2)^d + c z_0^d : z_1^d + c z_0^d].$$

For appropriate choices of the parameters c and λ, one can show that $\mathrm{supp}(T)$ and $\mathrm{supp}(\ )$ coincide. Moreover, f has an attracting fixed point, so the Fatou set is not empty. Observe that the restriction of f to the projective line $\{z_0 = 0\}$, for appropriate v, is chaotic, i.e. has dense orbits. One shows that $\{z_0 = 0\}$ is in the support of
  and that for appropriate m, $\mathbb{P}^2 \setminus \cup_{i=0}^m f^{-i}\{z_0 = 0\}$ is Kobayashi hyperbolic. Hence using the total invariance of $\mathrm{supp}(\ )$, we get that the complement of $\mathrm{supp}(\ )$ is in the Fatou set. It is possible to choose the parameters so that $\mathbb{P}^2 \setminus \mathrm{supp}(\ )$ contains an attractive fixed point. Several other examples are discussed in [FS6]. For example it is possible that $\mathrm{supp}(T)$ has non-empty interior and the Fatou set has also non-empty interior. The situation is then quite different from the one variable case, where either the Julia set is equal to $\mathbb{P}^1$ or it has empty interior.

We now give a characterization of the Julia sets in term of volume growth. There is an interesting gap in the possible volume growth.

Proposition 1.26. *Let f be a holomorphic endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$. Let T be its Green $(1,1)$-current. Then the following properties are equivalent:*

1. x is a point in the Julia set of order p, i.e. $x \in \mathscr{J}_p := \mathrm{supp}(T^p)$;
2. For every neighbourhood U of x, we have

$$\liminf_{n \to \infty} d^{-pn} \int_U (f^n)^*(\omega_{\mathrm{FS}}^p) \wedge \omega_{\mathrm{FS}}^{k-p} \neq 0;$$

3. For every neighbourhood U of x, we have

$$\limsup_{n \to \infty} d^{-(p-1)n} \int_U (f^n)^*(\omega_{\mathrm{FS}}^p) \wedge \omega_{\mathrm{FS}}^{k-p} = +\infty.$$

Proof. We have seen in Theorem 1.16 that $d^{-n}(f^n)^*(\omega_{\mathrm{FS}})$ converges to T when n goes to infinity. Moreover, $d^{-n}(f^n)^*(\omega_{\mathrm{FS}})$ admits a quasi-potential which converges uniformly to a quasi-potential of T. It follows that $\lim d^{-pn}(f^n)^*(\omega_{\mathrm{FS}}^p) = T^p$. We deduce that Properties 1) and 2) are equivalent. Since 2) implies 3), it remains to show that 3) implies 1). For this purpose, it is enough to show that for any open set V with $\overline{V} \cap \mathscr{J}_p = \varnothing$,

$$\int_V (f^n)^*(\omega_{\mathrm{FS}}^p) \wedge \omega_{\mathrm{FS}}^{k-p} = O(d^{(p-1)n}).$$

This is a consequence of a more general inequality in Theorem 1.112 below. We give here a direct proof.

Since $(\omega_{\mathrm{FS}} + dd^c \mathrm{g})^p = 0$ on $\mathbb{P}^k \setminus \mathscr{J}_p$, we can write there $\omega_{\mathrm{FS}}^p = dd^c \mathrm{g} \wedge (S^+ - S^-)$ where $S^{\pm}$ are positive closed $(p-1, p-1)$-currents on $\mathbb{P}^k$. Let χ be a cut-off function with compact support in $\mathbb{P}^k \setminus \mathscr{J}_p$ and equal to 1 on V. The above integral is bounded by

$$\int_{\mathbb{P}^k} \chi (f^n)^* \big(dd^c \mathrm{g} \wedge (S^+ - S^-)\big) \wedge \omega_{\mathrm{FS}}^{k-p} = \int_{\mathbb{P}^k} dd^c \chi \wedge (\mathrm{g} \circ f^n)(f^n)^*(S^+ - S^-) \wedge \omega_{\mathrm{FS}}^{k-p}.$$

Since g is bounded, the last integral is bounded by a constant times $\|(f^n)^*(S^+)\| + \|(f^n)^*(S^-)\|$. We conclude using the identity $\|(f^n)^*(S^{\pm})\| = d^{(p-1)n}\|S^{\pm}\|$. $\qquad \square$

The previous proposition suggests a notion of *local dynamical degree*. Define

$$\delta_p(x, r) := \limsup_{n \to \infty} \left(\int_{B(x,r)} (f^n)^* (\omega_{\mathrm{FS}}^p) \wedge \omega_{\mathrm{FS}}^{k-p} \right)^{1/n}$$

and

$$\delta_p(x) := \inf_{r > 0} \delta_p(x, r) = \lim_{r \to 0} \delta_p(x, r).$$

It follows from the above proposition that $\delta_p(x) = d^p$ for $x \in \mathscr{J}_p$ and $\delta_p(x) = 0$ for $x \notin \mathscr{J}_p$. This notion can be extended to general meromorphic maps or correspondences and the sub-level sets $\{\delta_p(x) \geq c\}$ can be seen as a kind of Julia sets.

Exercise 1.27. Let f be an endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$. Suppose a subsequence (f^{n_i}) is equicontinuous on an open set U. Show that U is contained in the Fatou set.

Exercise 1.28. Let f and g be two commuting holomorphic endomorphisms of $\mathbb{P}^k$, i.e. $f \circ g = g \circ f$. Show that f and g have the same Green currents. Deduce that they have the same Julia and Fatou sets.

Exercise 1.29. Determine the Green $(1,1)$-current and the Green measure for the map f in Example 1.11. Study the lamination on $\mathrm{supp}(T^p) \setminus \mathrm{supp}(T^{p+1})$. Express the current T^p on that set as an integral on appropriate manifolds.

Exercise 1.30. Let f be an endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$ and T its Green $(1,1)$-current. Consider the family of maps $\tau : \Delta \to \mathbb{P}^k$ such that $\tau^*(T) = 0$. The last equation means that if u is a local potential of T, i.e. $dd^c u = T$ on some open set, then $dd^c u \circ \tau = 0$ on its domain of definition. Show that the sequence $(f_{|\tau(\Delta)}^n)_{n \geq 1}$ is equicontinuous. Prove that there is a constant $c > 0$ such that $\|D\tau(0)\| \leq c$ for every τ as above (this property holds for any positive closed $(1,1)$-current T with continuous potentials). Find the corresponding discs for f as in Exercise 1.29.

Exercise 1.31. Let f be an endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$. Let X be an analytic set of pure dimension p in an open set $U \subset \mathbb{P}^k$. Show that for every compact $K \subset U$

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathrm{volume}(f^n(X \cap K)) \leq p \log d.$$

Hint. For an appropriate cut-off function χ, estimate $\int_X \chi (f^n)^* (\omega_{\mathrm{FS}}^p)$.

1.3 Other Constructions of the Green Currents

In this paragraph, we give other methods, introduced and developped by the authors, in order to construct the Green currents and Green measures for meromorphic maps. We obtain estimates on the Perron-Frobenius operator and on the thickness of the Green measure, that will be applied in the stochastic study of the dynamical system. A key point here is the use of d.s.h. functions as observables.

We first present a recent direct construction of Green (p,p)-currents using super-potentials[1]. Super-potentials are a tool in order to compute with positive closed (p,p)-currents. They play the same role as potentials for bidegree $(1,1)$ currents. In dynamics, they are used in particular in the equidistribution problem for algebraic sets of arbitrary dimension and allow to get some estimates on the speed of convergence.

Theorem 1.32. *Let S be a positive closed (p,p)-current of mass 1 on $\mathbb{P}^k$. Assume that the super-potential of S is bounded. Then $d^{-pn}(f^n)^*(S)$ converge to the Green (p,p)-current T^p of f. Moreover, T^p has a Hölder continuous super-potential.*

Sketch of proof. We refer to Appendix A.2 and A.4 for an introduction to super-potentials and to the action of holomorphic maps on positive closed currents. Recall that f^* and f_* act on $H^{p,p}(\mathbb{P}^k, \mathbb{C})$ as the multiplications by d^p and d^{k-p} respectively. So, if S is a positive closed (p,p)-current of mass 1, then $\|f^*(S)\| = d^p$ and $\|f_*(S)\| = d^{k-p}$ since the mass can be computed cohomologically. Let Λ denote the operator $d^{-p+1}f_*$ acting on $\mathscr{C}_{k-p+1}(\mathbb{P}^k)$, the convex set of positive closed currents of bidegree $(k-p+1, k-p+1)$ and of mass 1. It is continuous and it takes values also in $\mathscr{C}_{k-p+1}(\mathbb{P}^k)$. Let $\mathscr{V}$, $\mathscr{U}$, $\mathscr{U}_n$ denote the super-potentials of $d^{-p}f^*(\omega_{\mathrm{FS}}^p)$, S and $d^{-pn}(f^n)^*(S)$ respectively. Consider a quasi-potential U of mean 0 of S which is a DSH current satisfying $dd^c U = S - \omega_{\mathrm{FS}}^p$. The following computations are valid for S smooth and can be extended to all currents S using a suitable regularization procedure.

By Theorem A.35 in the Appendix, the current $d^{-p}f^*(U)$ is DSH and satisfies

$$dd^c \left(d^{-p} f^*(U) \right) = d^{-p} f^*(S) - d^{-p} f^*(\omega_{\mathrm{FS}}^p).$$

---

[1] These super-potentials correspond to super-potentials of mean 0 in [DS10].

If V is a smooth quasi-potential of mean 0 of $d^{-p}f^*(\omega_{FS}^p)$, i.e. a smooth real $(p-1,p-1)$-form such that

$$dd^cV = d^{-p}f^*(\omega_{FS}^p) - \omega_{FS}^p \quad \text{and} \quad \langle \omega_{FS}^{k-p+1}, V \rangle = 0,$$

then $V + d^{-p}f^*(U)$ is a quasi-potential of $d^{-p}f^*(S)$. Let m be the real number such that $V + d^{-p}f^*(U) + m\omega_{FS}^{p-1}$ is a quasi-potential of mean 0 of $d^{-1}f^*(S)$. We have

$$\begin{aligned}
\mathscr{U}_1(R) &= \langle V + d^{-p}f^*(U) + m\omega_{FS}^{p-1}, R \rangle \\
&= \langle V, R \rangle + d^{-1}\langle U, \Lambda(R) \rangle + m \\
&= \mathscr{V}(R) + d^{-1}\mathscr{U}(\Lambda(R)) + m.
\end{aligned}$$

By induction, we obtain

$$\begin{aligned}
\mathscr{U}_n(R) = {}& \mathscr{V}(R) + d^{-1}\mathscr{V}(\Lambda(R)) + \cdots + d^{-n+1}\mathscr{V}(\Lambda^{n-1}(R)) \\
&+ d^{-n}\mathscr{U}(\Lambda^n(R)) + m_n,
\end{aligned}$$

where m_n is a constant depending on n and on S.

Since $d^{-p}f^*(\omega_{FS}^p)$ is smooth, $\mathscr{V}$ is a Hölder continuous function. It is not difficult to show that Λ is Lipschitz with respect to the distance dist_α on $\mathscr{C}_{k-p+1}(\mathbb{P}^k)$. Therefore, by Lemma 1.19, the sum

$$\mathscr{V}(R) + d^{-1}\mathscr{V}(\Lambda(R)) + \cdots + d^{-n+1}\mathscr{V}(\Lambda^{n-1}(R))$$

converges uniformly to a Hölder continuous function $\mathscr{V}_\infty$ which does not depend on S. Recall that super-potentials vanish at ω_{FS}^{k-p+1}, in particular, $\mathscr{U}_n(\omega_{FS}^{k-p+1}) = 0$. Since $\mathscr{U}$ is bounded, the above expression of $\mathscr{U}_n(R)$ for $R = \omega_{FS}^{k-p+1}$ implies that m_n converge to $m_\infty := -\mathscr{V}_\infty(\omega_{FS}^{k-p+1})$ which is independent of S. So, $\mathscr{U}_n$ converge uniformly to $\mathscr{V}_\infty + m_\infty$. We deduce that $d^{-pn}(f^n)^*(S)$ converge to a current T_p which does not depend on S. Moreover, the super-potential of T_p is the Hölder continuous function $\mathscr{V}_\infty + m_\infty$.

We deduce from the above discussion that $d^{-pn}(f^n)^*(\omega_{FS}^p)$ converge in the Hartogs' sense to T_p. Theorem A.48 implies that $T_{p+q} = T_p \wedge T_q$ if $p + q \leq k$. Therefore, if T is the Green $(1,1)$-current, T_p is equal to T^p the Green (p,p)-current of f. $\qquad\square$

Remark 1.33. Let S_n be positive closed (p,p)-currents of mass 1 on $\mathbb{P}^k$. Assume that their super-potentials $\mathscr{U}_{S_n}$ satisfy $\|\mathscr{U}_{S_n}\|_\infty = o(d^n)$. Then $d^{-pn}(f^n)^*(S_n)$ converge to T^p. If (f_s) is a family of maps depending holomorphically on s in a space of parameters Σ, then the Green super-functions are also locally Hölder continuous with respect to s and define a positive closed (p,p)-current on $\Sigma \times \mathbb{P}^k$. Its slice by $\{s\} \times \mathbb{P}^k$ is the Green (p,p)-current of f_s.

We now introduce two other constructions of the Green measure. The main point is the use of appropriate spaces of test functions adapted to complex analysis. Their norms take into account the complex structure of $\mathbb{P}^k$. The reason to introduce these spaces is that they are invariant under the push-forward by a holomorphic map. This

is not the case for spaces of smooth forms because of the critical set. Moreover, we will see that there is a spectral gap for the action of endomorphisms of $\mathbb{P}^k$ which is a useful property in the stochastic study of the dynamical system. The first method, called *the dd^c-method*, was introduced in [DS1] and developed in [DS6]. It can be extended to Green currents of any bidegree. We show a convergence result for PB measures ν towards the Green measure. PB measures are diffuse in some sense; we will study equidistribution of Dirac masses in the next paragraph.

Recall that f is an endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$. Define the *Perron-Frobenius operator* Λ on test functions φ by $\Lambda(\varphi) := d^{-k} f_*(\varphi)$. More precisely, we have

$$\Lambda(\varphi)(z) := d^{-k} \sum_{w \in f^{-1}(z)} \varphi(w),$$

where the points in $f^{-1}(z)$ are counted with multiplicity. The following proposition is crucial.

Proposition 1.34. *The operator $\Lambda : \mathrm{DSH}(\mathbb{P}^k) \to \mathrm{DSH}(\mathbb{P}^k)$ is well-defined, bounded and continuous with respect to the weak topology on $\mathrm{DSH}(\mathbb{P}^k)$. The operator $\widetilde{\Lambda} : \mathrm{DSH}(\mathbb{P}^k) \to \mathrm{DSH}(\mathbb{P}^k)$ defined by*

$$\widetilde{\Lambda}(\varphi) := \Lambda(\varphi) - \langle \omega_{\mathrm{FS}}^k, \Lambda(\varphi) \rangle$$

is contracting and satisfies the estimate

$$\|\widetilde{\Lambda}(\varphi)\|_{\mathrm{DSH}} \leq d^{-1} \|\varphi\|_{\mathrm{DSH}}.$$

Proof. We prove the first assertion. Let φ be a quasi-p.s.h. function such that $dd^c \varphi \geq -\omega_{\mathrm{FS}}$. We show that $\Lambda(\varphi)$ is d.s.h. Since φ is strongly upper semi-continuous, $\Lambda(\varphi)$ is strongly upper semi-continuous, see Appendix A.2. If $dd^c \varphi = S - \omega_{\mathrm{FS}}$ with S positive closed, we have $dd^c \Lambda(\varphi) = d^{-k} f_*(S) - d^{-k} f_*(\omega_{\mathrm{FS}})$. Therefore, if u is a quasi-potential of $d^{-k} f_*(\omega_{\mathrm{FS}})$, then $u + \Lambda(\varphi)$ is strongly semi-continuous and is a quasi-potential of $d^{-k} f_*(S)$. So, this function is quasi-p.s.h. We deduce that $\Lambda(\varphi)$ is d.s.h., and hence $\Lambda : \mathrm{DSH}(\mathbb{P}^k) \to \mathrm{DSH}(\mathbb{P}^k)$ is well-defined.

Observe that $\Lambda : L^1(\mathbb{P}^k) \to L^1(\mathbb{P}^k)$ is continuous. Indeed, if φ is in $L^1(\mathbb{P}^k)$, we have

$$\|\Lambda(\varphi)\|_{L^1} = \langle \omega_{\mathrm{FS}}^k, d^{-k} |f_*(\varphi)| \rangle \leq \langle \omega_{\mathrm{FS}}^k, d^{-k} f_*(|\varphi|) \rangle = d^{-k} \langle f^*(\omega_{\mathrm{FS}}^k), |\varphi| \rangle \lesssim \|\varphi\|_{L^1}.$$

Therefore, $\Lambda : \mathrm{DSH}(\mathbb{P}^k) \to \mathrm{DSH}(\mathbb{P}^k)$ is continuous with respect to the weak topology. This and the estimates below imply that Λ is a bounded operator.

We prove now the last estimate in the proposition. Write $dd^c \varphi = S^+ - S^-$ with $S^\pm$ positive closed. We have

$$dd^c \widetilde{\Lambda}(\varphi) = dd^c \Lambda(\varphi) = d^{-k} f_*(S^+ - S^-) = d^{-k} f_*(S^+) - d^{-k} f_*(S^-).$$

Since $\|f_*(S^\pm)\| = d^{k-1} \|S^\pm\|$ and $\langle \omega_{\mathrm{FS}}^k, \widetilde{\Lambda}(\varphi) \rangle = 0$, we obtain that $\|\widetilde{\Lambda}(\varphi)\|_{\mathrm{DSH}} \leq d^{-1} \|S^\pm\|$. The result follows. $\square$

Recall that if ν is a positive measure on $\mathbb{P}^k$, the pull-back $f^*(\nu)$ is defined by the formula $\langle f^*(\nu), \varphi \rangle = \langle \nu, f_*(\varphi) \rangle$ for φ continuous on $\mathbb{P}^k$. Observe that since f is a ramified covering, $f_*(\varphi)$ is continuous when φ is continuous, see Exercise A.11 in Appendix. So, the above definition makes sense. For $\varphi = 1$, we obtain that $\|f^*(\nu)\| = d^k \|\nu\|$, since the covering is of degree d^k. If ν is the Dirac mass at a point a, $f^*(\nu)$ is the sum of Dirac masses at the points in $f^{-1}(a)$.

Recall that a measure ν is PB if quasi-p.s.h. are ν-integrable and ν is PC if it is PB and acts continuously on $\mathrm{DSH}(\mathbb{P}^k)$ with respect to the weak topology on this space, see Appendix A.4. We deduce from Proposition 1.34 the following result where the norm $\|\cdot\|$ on $\mathrm{DSH}(\mathbb{P}^k)$ is defined by

$$\|\varphi\| := |\langle\ , \varphi \rangle| + \inf \|S^\pm\|,$$

with $S^\pm$ positive closed such that $dd^c \varphi = S^+ - S^-$. We will see that   is PB, hence this norm is equivalent to $\|\cdot\|_{\mathrm{DSH}}$, see Proposition A.43.

Theorem 1.35. *Let f be as above. If ν is a PB probability measure, then $d^{-kn}(f^n)^*(\nu)$ converge to a PC probability measure   which is independent of ν and totally invariant under f. Moreover, if φ is a d.s.h. function and $c_\varphi := \langle\ , \varphi \rangle$, then*

$$\|\Lambda^n(\varphi) - c_\varphi\|\ \leq d^{-n}\|\varphi\| \quad \text{and} \quad \|\Lambda^n(\varphi) - c_\varphi\|_{\mathrm{DSH}} \leq A d^{-n}\|\varphi\|_{\mathrm{DSH}},$$

where $A > 0$ is a constant independent of φ and n. In particular, there is a constant $c > 0$ depending on ν such that

$$|\langle d^{-kn}(f^n)^*(\nu) -\ , \varphi \rangle| \leq c d^{-n}\|\varphi\|_{\mathrm{DSH}}.$$

Proof. Since ν is PB, d.s.h. functions are ν integrable. It follows that there is a constant $c > 0$ such that $|\langle \nu, \varphi \rangle| \leq c\|\varphi\|_{\mathrm{DSH}}$. Otherwise, there are d.s.h. functions φ_n with $\|\varphi_n\|_{\mathrm{DSH}} \leq 1$ and $\langle \nu, \varphi_n \rangle \geq 2^n$, hence the d.s.h. function $\sum 2^{-n}\varphi_n$ is not ν-integrable.

It follows from Proposition 1.34 that $f^*(\nu)$ is PB. So, $d^{-kn}(f^n)^*(\nu)$ is PB for every n. Define for φ d.s.h.,

$$c_0 := \langle \omega_{\mathrm{FS}}^k, \varphi \rangle \quad \text{and} \quad \varphi_0 := \varphi - c_0$$

and inductively

$$c_{n+1} := \langle \omega_{\mathrm{FS}}^k, \Lambda(\varphi_n) \rangle \quad \text{and} \quad \varphi_{n+1} := \Lambda(\varphi_n) - c_{n+1} = \widetilde{\Lambda}(\varphi_n).$$

A straighforward computation gives

$$\Lambda^n(\varphi) = c_0 + \cdots + c_n + \varphi_n.$$

Therefore,

$$\langle d^{-kn}(f^n)^*(\nu), \varphi \rangle = \langle \nu, \Lambda^n(\varphi) \rangle = c_0 + \cdots + c_n + \langle \nu, \varphi_n \rangle.$$

Proposition 1.34 applied inductively on n implies that $\|\varphi_n\|_{\mathrm{DSH}} \leq d^{-n}\|\varphi\|_{\mathrm{DSH}}$. Since Λ is bounded, it follows that $|c_n| \leq Ad^{-n}\|\varphi\|_{\mathrm{DSH}}$, where $A > 0$ is a constant. The property that ν is PB and the above estimate on φ_n imply that $\langle \nu, \varphi_n \rangle$ converge to 0.

We deduce that $\langle d^{-kn}(f^n)^*(\nu), \varphi \rangle$ converge to $c_\varphi := \sum_{n \geq 0} c_n$ and $|c_\varphi| \lesssim \|\varphi\|_{\mathrm{DSH}}$. Therefore, $d^{-kn}(f^n)^*(\nu)$ converge to a PB measure μ defined by $\langle \mu, \varphi \rangle := c_\varphi$. The constant c_φ does not depend on ν, hence the measure μ is independent of ν. The above convergence implies that μ is totally invariant, i.e. $f^*(\mu) = d^k \mu$. Finally, since c_n depends continuously on the d.s.h. function φ, the constant c_φ, which is defined by a normally convergent series, depends also continuously on φ. It follows that μ is PC.

We prove now the estimates in the theorem. The total invariance of μ implies that $\langle \mu, \Lambda^n(\varphi) \rangle = \langle \mu, \varphi \rangle = c_\varphi$. If $dd^c\varphi = S^+ - S^-$ with $S^\pm$ positive closed, we have $dd^c\Lambda^n(\varphi) = d^{-kn}(f^n)_*(S^+) - d^{-kn}(f^n)_*(S^-)$, hence

$$\|\Lambda^n(\varphi) - c_\varphi\|_* \leq d^{-kn}\|(f^n)_*(S^\pm)\| = d^{-n}\|S^\pm\|.$$

It follows that

$$\|\Lambda^n(\varphi) - c_\varphi\|_* \leq d^{-n}\|\varphi\|_*.$$

For the second estimate, we have

$$\|\Lambda^n(\varphi) - c_\varphi\|_{\mathrm{DSH}} = \|\varphi_n\|_{\mathrm{DSH}} + \sum_{i \geq n} c_i.$$

The last sum is clearly bounded by a constant times $d^{-n}\|\varphi\|_{\mathrm{DSH}}$. This together with the inequality $\|\varphi_n\|_{\mathrm{DSH}} \lesssim d^{-n}\|\varphi\|_{\mathrm{DSH}}$ implies $\|\Lambda^n(\varphi) - c_\varphi\|_{\mathrm{DSH}} \lesssim d^{-n}\|\varphi\|_{\mathrm{DSH}}$. We can also use that $\|\ \|_*$ and $\|\ \|_{\mathrm{DSH}}$ are equivalent, see Proposition A.43.

The last inequality in the theorem is then deduced from the identity

$$\langle d^{-kn}(f^n)^*(\nu) - \mu, \varphi \rangle = \langle \nu, \Lambda^n(\varphi) - c_\varphi \rangle$$

and the fact that ν is PB.   □

Remark 1.36. In the present case, the dd^c-method is quite simple. The function φ_n is the normalized solution of the equation $dd^c\psi = dd^c\Lambda^n(\varphi)$. It satisfies automatically good estimates. The other solutions differ from φ_n by constants. We will see that for polynomial-like maps, the solutions differ by pluriharmonic functions and the estimates are less straightforward. In the construction of Green (p, p)-currents with $p > 1$, φ is replaced with a test form of bidegree $(k - p, k - p)$ and φ_n is a solution of an appropriate dd^c-equation. The constants c_n will be replaced with dd^c-closed currents with a control of their cohomology class.

The second construction of the Green measure follows the same lines but we use the complex Sobolev space $W^*(\mathbb{P}^k)$ instead of $\mathrm{DSH}(\mathbb{P}^k)$. We obtain that the Green measure μ is WPB, see Appendix A.4 for the terminology. We only mention here the result which replaces Proposition 1.34.

Proposition 1.37. *The operator* $\Lambda : W^*(\mathbb{P}^k) \to W^*(\mathbb{P}^k)$ *is well-defined, bounded and continuous with respect to the weak topology on* $W^*(\mathbb{P}^k)$. *The operator* $\widetilde{\Lambda} : W^*(\mathbb{P}^k) \to W^*(\mathbb{P}^k)$ *defined by*

$$\widetilde{\Lambda}(\varphi) := \Lambda(\varphi) - \langle \omega_{\mathrm{FS}}^k, \Lambda(\varphi) \rangle$$

is contracting and satisfies the estimate

$$\|\widetilde{\Lambda}(\varphi)\|_{W^*} \le d^{-1/2} \|\varphi\|_{W^*}.$$

Sketch of proof. As in Proposition 1.34, since φ is in $L^1(\mathbb{P}^k)$, $\Lambda(\varphi)$ is also in $L^1(\mathbb{P}^k)$ and the main point here is to estimate $\partial\varphi$. Let S be a positive closed $(1,1)$-current on $\mathbb{P}^k$ such that $\sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \le S$. We show that $\sqrt{-1}\partial f_*(\varphi) \wedge \overline{\partial} f_*(\varphi) \le d^k f_*(S)$, in particular, the Poincaré differential $d\Lambda(\varphi)$ of $\Lambda(\varphi)$ is in $L^2(\mathbb{P}^k)$.

If a is not a critical value of f and U a small neighbourhood of a, then $f^{-1}(U)$ is the union of d^k open sets U_i which are sent bi-holomorphically on U. Let $g_i : U \to U_i$ be the inverse branches of f. On U, we obtain using Schwarz's inequality that

$$
\begin{aligned}
\sqrt{-1}\partial f_*(\varphi) \wedge \overline{\partial} f_*(\varphi) &= \sqrt{-1}\left(\sum \partial g_i^*(\varphi)\right) \wedge \left(\sum \overline{\partial} g_i^*(\varphi)\right) \\
&\le d^k \sum \sqrt{-1}\partial g_i^*(\varphi) \wedge \overline{\partial} g_i^*(\varphi) \\
&= d^k f_*\left(\sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi\right).
\end{aligned}
$$

Therefore, we have $\sqrt{-1}\partial f_*(\varphi) \wedge \overline{\partial} f_*(\varphi) \le d^k f_*(S)$ out of the critical values of f which is a manifold of real codimension 2.

Recall that $f_*(\varphi)$ is in $L^1(\mathbb{P}^k)$. It is a classical result in Sobolev spaces theory that an L^1 function whose gradient out of a submanifold of codimension 2 is in L^2, is in fact in the Sobolev space $W^{1,2}(\mathbb{P}^k)$. We deduce that the inequality $\sqrt{-1}\partial f_*(\varphi) \wedge \overline{\partial} f_*(\varphi) \le d^k f_*(S)$ holds on $\mathbb{P}^k$, because the left hand side term is an L^1 form and has no mass on critical values of f. Finally, we have

$$\sqrt{-1}\partial\Lambda(\varphi) \wedge \overline{\partial}\Lambda(\varphi) \le d^{-k} f_*(S).$$

This, together with the identity $\|f_*(S)\| = d^{k-1}\|S\|$, implies that $\|\widetilde{\Lambda}(\varphi)\|_{W^*} \le d^{-1/2}\|S\|$. The proposition follows. $\qquad\square$

In the rest of this paragraph, we show that the Green measure   is moderate, see Appendix A.4. Recall that a positive measure v on $\mathbb{P}^k$ is *moderate* if there are constants $\alpha > 0$ and $c > 0$ such that

$$\|e^{-\alpha\varphi}\|_{L^1(v)} \le c$$

for φ quasi-p.s.h. such that $dd^c\varphi \ge -\omega_{\mathrm{FS}}$ and $\langle \omega_{\mathrm{FS}}^k, \varphi \rangle = 0$. Moderate measures are PB and by linearity, if v is moderate and $\mathscr{D}$ is a bounded set of d.s.h. functions then there are constants $\alpha > 0$ and $c > 0$ such that

$$\|e^{\alpha|\varphi|}\|_{L^1(v)} \le c \quad \text{for} \quad \varphi \in \mathscr{D}.$$

Moderate measures were introduced in [DS1]. The fundamental estimate in Theorem A.22 in Appendix implies that smooth measures are moderate. So, when we use test d.s.h. functions, several estimates for the Lebesgue measure can be extended to moderate measures. For example, we will see that quasi-p.s.h. functions have comparable repartition functions with respect to the Lebesgue measure ω_{FS}^k and to the equilibrium measure .

It is shown in [DNS] that measures which are wedge-products of positive closed $(1,1)$-currents with Hölder continuous potentials, are moderate. In particular, the Green measure is moderate. We will give here another proof of this result using the following criterion. Since $\mathrm{DSH}(\mathbb{P}^k)$ is a subspace of $L^1(\mathbb{P}^k)$, the L^1-norm induces a distance on $\mathrm{DSH}(\mathbb{P}^k)$ that we denote by dist_{L^1}.

Proposition 1.38. *Let ν be a PB positive measure on $\mathbb{P}^k$. Assume that ν restricted to any bounded subset of $\mathrm{DSH}(\mathbb{P}^k)$ is Hölder continuous[2] with respect to dist_{L^1}. Then ν is moderate.*

Proof. Let φ be a quasi-p.s.h. function such that $dd^c\varphi \geq -\omega_{\mathrm{FS}}$ and $\langle \omega_{\mathrm{FS}}^k, \varphi \rangle = 0$. We want to prove that $\langle \nu, e^{-\alpha\varphi} \rangle \leq c$ for some positive constants α, c. For this purpose, we only have to show that $\nu\{\varphi \leq -M\} \lesssim e^{-\alpha M}$ for some constant $\alpha > 0$ and for $M \geq 1$. Define for $M > 0$, $\varphi_M := \max(\varphi, -M)$. These functions φ_M belong to a compact family $\mathscr{D}$ of d.s.h. functions. Observe that $\varphi_{M-1} - \varphi_M$ is positive with support in $\{\varphi \leq -M + 1\}$. It is bounded by 1 and equal to 1 on $\{\varphi < -M\}$. Therefore, the Hölder continuity of ν on $\mathscr{D}$ implies that there is a constant $\lambda > 0$ such that

$$\nu\{\varphi < -M\} \leq \langle \nu, \varphi_{M-1} - \varphi_M \rangle = \nu(\varphi_{M-1}) - \nu(\varphi_M)$$
$$\lesssim \mathrm{dist}_{L^1}(\varphi_{M-1}, \varphi_M)^\lambda \leq \mathrm{volume}\{\varphi < -M + 1\}^\lambda.$$

Since the Lebesgue measure is moderate, the last expression is $\lesssim e^{-\alpha M}$ for some positive constant α. The proposition follows. $\square$

We have the following result obtained in [DNS]. It will be used to establish several stochastic properties of d.s.h. observables for the equilibrium measure.

Theorem 1.39. *Let f be an endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$. Then, the Green measure of f is Hölder continuous on bounded subsets of $\mathrm{DSH}(\mathbb{P}^k)$. In particular, it is moderate.*

Proof. Let $\mathscr{D}$ be a bounded set of d.s.h. functions. We have to show that   is Hölder continuous on $\mathscr{D}$ with respect to dist_{L^1}. By linearity, since   is PC, it is enough to consider the case where $\mathscr{D}$ is the set of d.s.h. functions φ such that $\langle \ , \varphi \rangle \geq 0$ and $\|\varphi\| \leq 1$. Let $\widetilde{\mathscr{D}}$ denote the set of d.s.h. functions $\varphi - \Lambda(\varphi)$ with $\varphi \in \mathscr{D}$. By Proposition 1.34, $\widetilde{\mathscr{D}}$ is a bounded family of d.s.h. functions. We claim that $\widetilde{\mathscr{D}}$ is invariant under $\widetilde{\Lambda} := d\Lambda$. Observe that if φ is in $\mathscr{D}$, then $\widetilde{\varphi} := \varphi - \langle \ , \varphi \rangle$ is also in $\mathscr{D}$. Since $\langle \ , \varphi \rangle = \langle \ , \Lambda(\varphi) \rangle$, we have $\|\widetilde{\Lambda}(\widetilde{\varphi})\| \leq 1$ and

$$\widetilde{\Lambda}(\varphi - \Lambda(\varphi)) = \widetilde{\Lambda}(\widetilde{\varphi} - \Lambda(\widetilde{\varphi})) = \widetilde{\Lambda}(\widetilde{\varphi}) - \Lambda(\widetilde{\Lambda}(\widetilde{\varphi})).$$

---

[2] This property is close to the property that ν has a Hölder continuous super-potential.

By Theorem 1.35, $\widetilde{\Lambda}(\widetilde{\varphi})$ belongs to $\mathscr{D}$. This proves the claim. So, the crucial point is that Λ is contracting on an appropriate hyperplane.

For φ, ψ in $L^1(\mathbb{P}^k)$ we have

$$\|\widetilde{\Lambda}(\varphi) - \widetilde{\Lambda}(\psi)\|_{L^1} \leq \int \widetilde{\Lambda}(|\varphi - \psi|)\omega_{\mathrm{FS}}^k = d^{1-k}\int |\varphi - \psi||f^*(\omega_{\mathrm{FS}}^k) \lesssim \|\varphi - \psi\|_{L^1}.$$

So, the map $\widetilde{\Lambda}$ is Lipschitz with respect to dist_{L^1}. In particular, the map $\varphi \mapsto \varphi - \Lambda(\varphi)$ is Lipschitz with respect to this distance. Now, we have for $\varphi \in \mathscr{D}$

$$\langle\ ,\varphi\rangle = \lim_{n\to\infty}\langle d^{-kn}(f^n)^*(\omega_{\mathrm{FS}}^k),\varphi\rangle = \lim_{n\to\infty}\langle\omega_{\mathrm{FS}}^k,\Lambda^n(\varphi)\rangle$$

$$= \langle\omega_{\mathrm{FS}}^k,\varphi\rangle - \sum_{n\geq 0}\langle\omega_{\mathrm{FS}}^k,\Lambda^n(\varphi) - \Lambda^{n+1}(\varphi)\rangle$$

$$= \langle\omega_{\mathrm{FS}}^k,\varphi\rangle - \sum_{n\geq 0}d^{-n}\langle\omega_{\mathrm{FS}}^k,\widetilde{\Lambda}^n(\varphi - \Lambda(\varphi))\rangle.$$

By Lemma 1.19, the last series defines a function on $\widetilde{\widetilde{\mathscr{D}}}$ which is Hölder continuous with respect to dist_{L^1}. Therefore, $\varphi \mapsto \langle\ ,\varphi\rangle$ is Hölder continuous on $\mathscr{D}$. $\qquad\square$

Remark 1.40. Let f_s be a family of endomorphisms of algebraic degree $d \geq 2$, depending holomorphically on a parameter $s \in \Sigma$. Let $\ _s$ denote its equilibrium measure. We get that $(s,\varphi) \mapsto\ _s(\varphi)$ is Hölder continuous on bounded subsets of $\Sigma \times \mathrm{DSH}(\mathbb{P}^k)$.

The following results are useful in the stochastic study of the dynamical system.

Corollary 1.41. *Let $f,\ $ and Λ be as above. There are constants $c > 0$ and $\alpha > 0$ such that if ψ is d.s.h. with $\|\psi\|_{\mathrm{DSH}} \leq 1$, then*

$$\langle\ ,e^{\alpha d^n|\Lambda^n(\psi)-\langle\ ,\psi\rangle|}\rangle \leq c \quad and \quad \|\Lambda^n(\psi) - \langle\ ,\psi\rangle\|_{L^q(\ )} \leq cqd^{-n}$$

for every $n \geq 0$ and every $1 \leq q < +\infty$.

Proof. By Theorem 1.35, $d^n(\Lambda^n(\psi) - \langle\ ,\psi\rangle)$ belong to a compact family in $\mathrm{DSH}(\mathbb{P}^k)$. The first inequality in the corollary follows from Theorem 1.39. For the second one, we can assume that q is integer and we easily deduce the result from the first inequality and the inequalities $x^q \leq q!e^x \leq q^q e^x$ for $x \geq 0$. $\qquad\square$

Corollary 1.42. *Let $0 < \nu \leq 2$ be a constant. There are constants $c > 0$ and $\alpha > 0$ such that if ψ is a ν-Hölder continuous function with $\|\psi\|_{\mathscr{C}^\nu} \leq 1$, then*

$$\langle\ ,e^{\alpha d^{n\nu/2}|\Lambda^n(\psi)-\langle\ ,\psi\rangle|}\rangle \leq c \quad and \quad \|\Lambda^n(\psi) - \langle\ ,\psi\rangle\|_{L^q(\ )} \leq cq^{\nu/2}d^{-n\nu/2}$$

for every $n \geq 0$ and every $1 \leq q < +\infty$.

Proof. By Corollary 1.41, since $\|\cdot\|_{\mathrm{DSH}} \lesssim \|\cdot\|_{\mathscr{C}^2}$, we have

$$\|\Lambda^n(\psi) - \langle\ ,\psi\rangle\|_{L^q(\ )} \leq cqd^{-n}\|\psi\|_{\mathscr{C}^2},$$

with $c > 0$ independent of q and of ψ. On the other hand, by definition of Λ, we have

$$\|\Lambda^n(\psi) - \langle \ , \psi \rangle\|_{L^q(\ )} \leq \|\Lambda^n(\psi) - \langle \ , \psi \rangle\|_{L^\infty(\ )} \leq 2\|\psi\|_{\mathscr{C}^0}.$$

The theory of interpolation between the Banach spaces $\mathscr{C}^0$ and $\mathscr{C}^2$ [T1], applied to the linear operator $\psi \mapsto \Lambda^n(\psi) - \langle \ , \psi \rangle$, implies that

$$\|\Lambda^n(\psi) - \langle \ , \psi \rangle\|_{L^q(\ )} \leq A_\nu 2^{1-\nu/2}[cqd^{-n}]^{\nu/2}\|\psi\|_{\mathscr{C}^\nu},$$

for some constant $A_\nu > 0$ depending only on ν and on $\mathbb{P}^k$. This gives the second inequality in the corollary.

Recall that if L is a linear continuous functional on the space $\mathscr{C}^0$ of continuous functions, then we have for every $0 < \nu < 2$

$$\|L\|_{\mathscr{C}^\nu} \leq A_\nu \|L\|_{\mathscr{C}^0}^{1-\nu/2} \|L\|_{\mathscr{C}^2}^{\nu/2}$$

for some constant $A_\nu > 0$ independent of L (in our case, the functional is with values in $L^q(\ )$).

For the first inequality, we have for a fixed constant $\alpha > 0$ small enough,

$$\langle \ , e^{\alpha d^{n\nu/2}|\Lambda^n(\psi) - \langle \ ,\psi \rangle|} \rangle = \sum_{q \geq 0} \frac{1}{q!} \langle \ , |\alpha d^{n\nu/2}(\Lambda^n(\psi) - \langle \ , \psi \rangle)|^q \rangle \leq \sum_{q \geq 0} \frac{1}{q!} \alpha^q c^q q^q.$$

By Stirling's formula, the last sum converges. The result follows.   □

Exercise 1.43. Let φ be a smooth function and φ_n as in Theorem 1.35. Show that we can write $\varphi_n = \varphi_n^+ - \varphi_n^-$ with $\varphi_n^\pm$ quasi-p.s.h. such that $\|\varphi_n^\pm\|_{\mathrm{DSH}} \lesssim d^{-n}$ and $dd^c \varphi_n^\pm \gtrsim -d^{-n}\omega_{\mathrm{FS}}$. Prove that φ_n converge pointwise to 0 out of a pluripolar set. Deduce that if ν is a probability measure with no mass on pluripolar sets, then $d^{-kn}(f^n)^*(\nu)$ converge to .

Exercise 1.44. Let $\mathrm{DSH}_0(\mathbb{P}^k)$ be the space of d.s.h. functions φ such that $\langle \ , \varphi \rangle = 0$. Show that $\mathrm{DSH}_0(\mathbb{P}^k)$ is a closed subspace of $\mathrm{DSH}(\mathbb{P}^k)$, invariant under Λ, and that the spectral radius of Λ on this space is equal to $1/d$. Note that 1 is an eigenvalue of Λ on $\mathrm{DSH}(\mathbb{P}^k)$, so, Λ has a spectral gap on $\mathrm{DSH}(\mathbb{P}^k)$. Prove a similar result for $W^*(\mathbb{P}^k)$.

1.4 Equidistribution of Points

In this paragraph, we show that the preimages of a generic point by f^n are equidistributed with respect to the Green measure   when n goes to infinity. The proof splits in two parts. First, we prove that there is a maximal proper algebraic set $\mathscr{E}$ which is totally invariant, then we show that for $a \notin \mathscr{E}$, the preimages of a are equidistributed. We will also prove that the convex set of probability measures ν,

which are totally invariant, i.e. $f^*(\nu) = d^k \nu$, is finite dimensional. The equidistribution for a out of an algebraic set is reminiscent of the main questions in value distribution theory (we will see in the next paragraph that using super-potentials we can get an estimate on the speed of convergence towards , at least for generic maps). Finally, we prove a theorem due to Briend-Duval on the equidistribution of the repelling periodic points. The following result was obtained by the authors in [DS1], see also [DS9] and [BH, FL, LY] for the case of dimension 1.

Theorem 1.45. *Let f be an endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$ and   its Green measure. Then there is a proper algebraic set $\mathscr{E}$ of $\mathbb{P}^k$, possibly empty, such that $d^{-kn}(f^n)^*(\delta_a)$ converge to   if and only if $a \notin \mathscr{E}$. Here, δ_a denotes the Dirac mass at a. Moreover, $\mathscr{E}$ is totally invariant: $f^{-1}(\mathscr{E}) = f(\mathscr{E}) = \mathscr{E}$ and is maximal in the sense that if E is a proper algebraic set of $\mathbb{P}^k$ such that $f^{-n}(E) \subset E$ for some $n \geq 1$, then E is contained in $\mathscr{E}$.*

Briend-Duval proved in [BD2] the above convergence for a outside the orbit of the critical set. They announced the property for a out of an algebraic set, but there is a problem with the counting of multiplicity in their lemma in [BD2, p.149].

We also have the following earlier result due to Fornæss-Sibony [FS1].

Proposition 1.46. *There is a pluripolar set $\mathscr{E}'$ such that if a is out of $\mathscr{E}'$, then $d^{-kn}(f^n)^*(\delta_a)$ converge to .*

Sketch of proof. We use here a version of the above dd^c-method which is given in [DS6] in a more general setting. Let φ be a smooth function and φ_n as in Theorem 1.35. Then, the functions φ_n are continuous. The estimates on φ_n imply that the series $\sum \varphi_n$ converges in DSH($\mathbb{P}^k$), hence converges pointwise out of a pluripolar set. Therefore, $\varphi_n(a)$ converge to 0 for a out of some pluripolar set E_φ, see Exercise 1.43. If $c_n := \langle \omega_{\mathrm{FS}}^k, \Lambda(\varphi_n) \rangle$, we have as in Theorem 1.35

$$\langle d^{-kn}(f^n)^*(\delta_a), \varphi \rangle = c_0 + \cdots + c_n + \langle \delta_a, \varphi_n \rangle = c_0 + \cdots + c_n + \varphi_n(a).$$

Therefore, $\langle d^{-kn}(f^n)^*(\delta_a), \varphi \rangle$ converge to $c_\varphi = \langle , \varphi \rangle$, for a out of E_φ.

Now, consider φ in a countable family $\mathscr{F}$ which is dense in the space of smooth functions. If a is not in the union $\mathscr{E}'$ of the pluripolar sets E_φ, the above convergence of $\langle d^{-kn}(f^n)^*(\delta_a), \varphi \rangle$ together with the density of $\mathscr{F}$ implies that $d^{-kn}(f^n)^*(\delta_a)$ converge to . Finally, $\mathscr{E}'$ is pluripolar since it is a countable union of such sets. $\square$

For the rest of the proof, we follow a geometric method introduced by Lyubich [LY] in dimension one and developed in higher dimension by Briend-Duval and Dinh-Sibony. We first prove the existence of the exceptional set and give several characterizations in the following general situation. Let X be an analytic set of pure dimension p in $\mathbb{P}^k$ invariant under f, i.e. $f(X) = X$. Let $g : X \to X$ denote the restriction of f to X. The following result can be deduced from Section 3.4 in [DS1], see also [D5, DS9].

Theorem 1.47. *There is a proper analytic set $\mathscr{E}_X$ of X, possibly empty, totally invariant under g, which is maximal in the following sense. If E is an analytic set of X, of*

dimension $< \dim X$, *such that* $g^{-s}(E) \subset E$ *for some* $s \geq 1$, *then* $E \subset \mathscr{E}_X$. *Moreover, there are at most finitely many analytic sets in* X *which are totally invariant under* g.

Since g permutes the irreducible components of X, we can find an integer $m \geq 1$ such that g^m fixes the components of X.

Lemma 1.48. *The topological degree of* g^m *is equal to* d^{mp}. *More precisely, there is a hypersurface* Y *of* X *containing* $\mathrm{sing}(X) \cup g^m(\mathrm{sing}(X))$ *such that for* $x \in X$ *out of* Y, *the fiber* $g^{-m}(x)$ *has exactly* d^{mp} *points.*

Proof. Since g^m fixes the components of X, we can assume that X is irreducible. It follows that g^m defines a covering over some Zariski dense open set of X. We want to prove that δ, the degree of this covering, is equal to d^{mp}. Consider the positive measure $(f^m)^*(\omega_{\mathrm{FS}}^p) \wedge [X]$. Since $(f^m)^*(\omega_{\mathrm{FS}}^p)$ is cohomologous to $d^{mp}\omega_{\mathrm{FS}}^p$, this measure is of mass $d^{mp} \deg(X)$. Observe that $(f^m)_*$ preserves the mass of positive measures and that we have $(f^m)_*[X] = \delta[X]$. Hence,

$$d^{mp} \deg(X) = \|(f^m)^*(\omega_{\mathrm{FS}}^p) \wedge [X]\| = \|(f^m)_*((f^m)^*(\omega_{\mathrm{FS}}^p) \wedge [X])\|$$
$$= \|\omega_{\mathrm{FS}}^p \wedge (f^m)_*[X]\| = \delta \|\omega_{\mathrm{FS}}^p \wedge [X]\| = \delta \deg(X).$$

It follows that $\delta = d^{mp}$. So, we can take for Y, a hypersurface which contains the ramification values of g^m and the set $\mathrm{sing}(X) \cup g^m(\mathrm{sing}(X))$. $\qquad\square$

Let Y be as above. Observe that if $g^m(x) \notin Y$ then g^m defines a bi-holomorphic map between a neighbourhood of x and a neighbourhood of $g^m(x)$ in X. Let $[Y]$ denote the $(k-p+1, k-p+1)$-current of integration on Y in $\mathbb{P}^k$. Since $(f^{mn})_*[Y]$ is a positive closed $(k-p+1, k-p+1)$-current of mass $d^{mn(p-1)} \deg(Y)$, we can define the following ramification current

$$R = \sum_{n \geq 0} R_n := \sum_{n \geq 0} d^{-mnp} (f^{mn})_*[Y].$$

Let $\nu(R, x)$ denote the Lelong number of R at x. By Theorem A.14, for $c > 0$, $E_c := \{\nu(R, x) \geq c\}$ is an analytic set of dimension $\leq p - 1$ contained in X. Observe that E_1 contains Y. We will see that R measures the obstruction for constructing good backwards orbits.

For any point $x \in X$ let $\lambda_n'(x)$ denote the number of distinct orbits

$$x_{-n}, x_{-n+1}, \ldots, x_{-1}, x_0$$

such that $g^m(x_{-i-1}) = x_{-i}$, $x_0 = x$ and $x_{-i} \in X \setminus Y$ for $0 \leq i \leq n-1$. These are the "good" orbits. Define $\lambda_n := d^{-mpn}\lambda_n'$. The function λ_n is lower semi-continuous with respect to the Zariski topology on X. Moreover, by Lemma 1.48, we have $0 \leq \lambda_n \leq 1$ and $\lambda_n = 1$ out of the analytic set $\cup_{i=0}^{n-1} g^{mi}(Y)$. The sequence (λ_n) decreases to a function λ, which represents the asymptotic proportion of backwards orbits in $X \setminus Y$.

Lemma 1.49. *There is a constant* $\gamma > 0$ *such that* $\lambda \geq \gamma$ *on* $X \setminus E_1$.

Proof. We deduce from Theorem A.14, the existence of a constant $0 < \gamma < 1$ satisfying $\{v(R,x) > 1 - \gamma\} = E_1$. Indeed, the sequence of analytic sets $\{v(R,x) \geq 1 - 1/i\}$ is decreasing, hence stationary. Consider a point $x \in X \setminus E_1$. We have $x \notin Y$ and if $v_n := v(R_n, x)$, then $\sum v_n \leq 1 - \gamma$. Since E_1 contains Y, $v_0 = 0$ and $F_1 := g^{-m}(x)$ contains exactly d^{mp} points. The definition of v_1, which is "the multiplicity" of $d^{-mp}(f^m)_*[Y]$ at x, implies that $g^{-m}(x)$ contains at most $v_1 d^{mp}$ points in Y. Then

$$\# g^{-m}(F_1 \setminus Y) = d^{mp} \#(F_1 \setminus Y) \geq (1 - v_1) d^{2mp}.$$

Define $F_2 := g^{-m}(F_1 \setminus Y)$. The definition of v_2 implies that F_2 contains at most $v_2 d^{2mp}$ points in Y. Hence, $F_3 := g^{-m}(F_2 \setminus Y)$ contains at least $(1 - v_1 - v_2) d^{3mp}$ points. In the same way, we define $F_4, \ldots, F_n$ with $\#F_n \geq (1 - \sum v_i) d^{mpn}$. Hence, for every n we get the following estimate:

$$\lambda_n(x) \geq d^{-mpn} \# F_n \geq 1 - \sum v_i \geq \gamma.$$

This proves the lemma.   □

End of the proof of Theorem 1.47. Let $\mathscr{E}_X^n$ denote the set of $x \in X$ such that $g^{-ml}(x) \subset E_1$ for $0 \leq l \leq n$ and define $\mathscr{E}_X := \cap_{n \geq 0} \mathscr{E}_X^n$. Then, $(\mathscr{E}_X^n)$ is a decreasing sequence of analytic subsets of E_1. It should be stationary. So, there is $n_0 \geq 0$ such that $\mathscr{E}_X^n = \mathscr{E}_X$ for $n \geq n_0$.

By definition, $\mathscr{E}_X$ is the set of $x \in X$ such that $g^{-mn}(x) \subset E_1$ for every $n \geq 0$. Hence, $g^{-m}(\mathscr{E}_X) \subset \mathscr{E}_X$. It follows that the sequence of analytic sets $g^{-mn}(\mathscr{E}_X)$ is decreasing and there is $n \geq 0$ such that $g^{-m(n+1)}(\mathscr{E}_X) = g^{-mn}(\mathscr{E}_X)$. Since g^{mn} is surjective, we deduce that $g^{-m}(\mathscr{E}_X) = \mathscr{E}_X$ and hence $\mathscr{E}_X = g^m(\mathscr{E}_X)$.

Assume as in the theorem that E is analytic with $g^{-s}(E) \subset E$. Define $E' := g^{-s+1}(E) \cup \ldots \cup E$. We have $g^{-1}(E') \subset E'$ which implies $g^{-n-1}(E') \subset g^{-n}(E')$ for every $n \geq 0$. Hence, $g^{-n-1}(E') = g^{-n}(E')$ for n large enough. This and the surjectivity of g imply that $g^{-1}(E') = g(E') = E'$. By Lemma 1.48, the topological degree of $(g^{m'})_{|E'}$ is at most $d^{m'(p-1)}$ for some $m' \geq 1$. This, the identity $g^{-1}(E') = g(E') = E'$ together with Lemma 1.49 imply that $E' \subset E_1$. Hence, $E' \subset \mathscr{E}_X$ and $E \subset \mathscr{E}_X$.

Define $\mathscr{E}_X' := g^{-m+1}(\mathscr{E}_X) \cup \ldots \cup \mathscr{E}_X$. We have $g^{-1}(\mathscr{E}_X') = g(\mathscr{E}_X') = \mathscr{E}_X'$. Applying the previous assertion to $E := \mathscr{E}_X'$ yields $\mathscr{E}_X' \subset \mathscr{E}_X$. Therefore, $\mathscr{E}_X' = \mathscr{E}_X$ and $g^{-1}(\mathscr{E}_X) = g(\mathscr{E}_X) = \mathscr{E}_X$. So, $\mathscr{E}_X$ is the maximal proper analytic set in X which is totally invariant under g.

We prove now that there are only finitely many totally invariant algebraic sets. We only have to consider totally invariant sets E of pure dimension q. The proof is by induction on the dimension p of X. The case $p = 0$ is clear. Assume that the assertion is true for X of dimension $\leq p - 1$ and consider the case of dimension p. If $q = p$ then E is a union of components of X. There are only a finite number of such analytic sets. If $q < p$, we have seen that E is contained in $\mathscr{E}_X$. Applying the induction hypothesis to the restriction of f to $\mathscr{E}_X$ gives the result.   □

We now give another characterization of $\mathscr{E}_X$. Observe that if X is not locally irreducible at a point x then $g^{-m}(x)$ may contain more than d^{mp} points. Let $\pi : \widetilde{X} \to X$

be the normalization of X, see Appendix A.1. By Theorem A.4 applied to $g \circ \pi$, g can be lifted to a map $\widetilde{g} : \widetilde{X} \to \widetilde{X}$ such that $g \circ \pi = \pi \circ \widetilde{g}$. Since g is finite, $\widetilde{g}$ is also finite. We deduce that $\widetilde{g}^m$ defines ramified coverings of degree d^{mp} on each component of $\widetilde{X}$. In particular, any fiber of $\widetilde{g}^m$ contains at most d^{mp} points. Observe that if $g^{-1}(E) \subset E$ then $\widetilde{g}^{-1}(\widetilde{E}) \subset \widetilde{E}$ where $\widetilde{E} := \pi^{-1}(E)$. Theorem 1.47 can be extended to $\widetilde{g}$. For simplicity, we consider the case where X is itself a normal analytic space. If X is not normal, one should work with its normalization.

Let Z be a hypersurface of X containing E_1. Let $N_n(a)$ denote the number of orbits of g^m

$$a_{-n}, \ldots, a_{-1}, a_0$$

with $g^m(a_{-i-1}) = a_{-i}$ and $a_0 = a$ such that $a_{-i} \in Z$ for every i. Here, the orbits are counted with multiplicity. So, $N_n(a)$ is the number of negative orbits of order n of a which stay in Z. Observe that the sequence of functions $\tau_n := d^{-pmn}N_n$ decreases to some function τ. Since τ_n are upper semi-continuous with respect to the Zariski topology and $0 \leq \tau_n \leq 1$ (we use here the assumption that X is normal), the function τ satisfies the same properties. Note that $\tau(a)$ is the probability that an infinite negative orbit of a stays in Z.

Proposition 1.50. *Assume that X is normal. Then, τ is the characteristic function of $\mathscr{E}_X$, that is, $\tau = 1$ on $\mathscr{E}_X$ and $\tau = 0$ on $X \setminus \mathscr{E}_X$.*

Proof. Since $\mathscr{E}_X \subset Z$ and $\mathscr{E}_X$ is totally invariant by g, we have $\mathscr{E}_X \subset \{\tau = 1\}$. Let $\theta \geq 0$ denote the maximal value of τ on $X \setminus \mathscr{E}_X$. This value exists since τ is upper semi-continuous with respect to the Zariski topology (indeed, it is enough to consider the algebraic subsets $\{\tau \geq \theta'\}$ of X which decrease when θ' increases; the family is stationary). We have to check that $\theta = 0$. Assume in order to obtain a contradiction that $\theta > 0$. Since $\tau \leq 1$, we always have $\theta \leq 1$. Consider the non-empty analytic set $E := \tau^{-1}(\theta) \setminus \mathscr{E}_X$ in $Z \setminus \mathscr{E}_X$. Let a be a point in E. Since $\mathscr{E}_X$ is totally invariant, we have $g^{-m}(a) \cap \mathscr{E}_X = \varnothing$. Hence, $\tau(b) \leq \theta$ for every $b \in g^{-m}(a)$. We deduce from the definition of τ and θ that

$$\theta = \tau(a) \leq d^{-pm} \sum_{b \in g^{-m}(a)} \tau(b) \leq \theta.$$

It follows that $g^{-m}(a) \subset E$. Therefore, the analytic subset $\overline{E}$ of Z satisfies $g^{-m}(\overline{E}) \subset \overline{E}$. This contradicts the maximality of $\mathscr{E}_X$. $\qquad\square$

We continue the proof of Theorem 1.45. We will use the above results for $X = \mathbb{P}^k$, Y the set of critical values of f. Let R be the ramification current defined as above by

$$R = \sum_{n \geq 0} R_n := \sum_{n \geq 0} d^{-kn}(f^n)_*[Y].$$

The following proposition was obtained in [DS1], a weaker version was independently obtained by Briend-Duval [BD3]. Here, an inverse branch on B for f^n is a bi-holomorphic map $g_i : B \to U_i$ such that $g_i \circ f^n$ is identity on U_i.

Proposition 1.51. *Let v be a strictly positive constant. Let a be a point in $\mathbb{P}^k$ such that the Lelong number $v(R,a)$ of R at a is strictly smaller than v. Then, there is a ball B centered at a such that f^n admits at least $(1-\sqrt{v})d^{kn}$ inverse branches $g_i : B \to U_i$ where U_i are open sets in $\mathbb{P}^k$ of diameter $\leq d^{-n/2}$. In particular, if $'$ is a limit value of the measures $d^{-kn}(f^n)^*(\delta_a)$ then $\| ' - \| \leq 2\sqrt{v(R,a)}$.*

Given a local coordinate system at a, let $\mathscr{F}$ denote the family of complex lines passing through a. For such a line Δ denote by Δ_r the disc of center a and of radius r. The family $\mathscr{F}$ is parametrized by $\mathbb{P}^{k-1}$ where the probability measure (the volume form) associated to the Fubini-Study metric is denoted by $\mathscr{L}$. Let B_r denote the ball of center a and of radius r.

Lemma 1.52. *Let S be a positive closed $(1,1)$-current on a neighbourhood of a. Then for any $\delta > 0$ there is an $r > 0$ and a family $\mathscr{F}' \subset \mathscr{F}$, such that $\mathscr{L}(\mathscr{F}') \geq 1 - \delta$ and for every Δ in $\mathscr{F}'$, the measure $S \wedge [\Delta_r]$ is well-defined and of mass $\leq v(S,a) + \delta$, where $v(S,a)$ is the Lelong number of S at a.*

Proof. Let $\pi : \widehat{\mathbb{P}}^k \to \mathbb{P}^k$ be the blow-up of $\mathbb{P}^k$ at a and E the exceptional hypersurface. Then, we can write $\pi^*(S) = v(S,a)[E] + S'$ with S' a current having no mass on E, see Exercise A.39. It is clear that for almost every Δ_r, the restriction of the potentials of S to Δ_r is not identically $-\infty$, so, the measure $S \wedge [\Delta_r]$ is well-defined. Let $\widehat{\Delta}_r$ denote the strict transform of Δ_r by π, i.e. the closure of $\pi^{-1}(\Delta_r \setminus \{a\})$. Then, the $\widehat{\Delta}_r$ define a smooth holomorphic fibration over E. The measure $S \wedge [\Delta_r]$ is equal to the push-forward of $\pi^*(S) \wedge [\widehat{\Delta}_r]$ by π. Observe that $\pi^*(S) \wedge [\widehat{\Delta}_r]$ is equal to $S' \wedge [\widehat{\Delta}_r]$ plus $v(S,a)$ times the Dirac mass at $\widehat{\Delta}_r \cap E$. Therefore, we only have to consider the Δ_r such that $S' \wedge [\widehat{\Delta}_r]$ are of mass $\leq \delta$.

Since S' have no mass on E, its mass on $\pi^{-1}(B_r)$ tends to 0 when r tends to 0. It follows from Fubini's theorem that when r is small enough the mass of the slices $S' \wedge [\widehat{\Delta}_r]$ is $\leq \delta$ except for a small family of Δ. This proves the lemma. $\qquad\square$

Lemma 1.53. *Let U be a neighbourhood of $\overline{B}_r$. Let S be a positive closed $(1,1)$-current on U. Then, for every $\delta > 0$, there is a family $\mathscr{F}' \subset \mathscr{F}$ with $\mathscr{L}(\mathscr{F}') > 1 - \delta$, such that for Δ in $\mathscr{F}'$, the measure $S \wedge [\Delta_r]$ is well-defined and of mass $\leq A\|S\|$, where $A > 0$ is a constant depending on δ but independent of S.*

Proof. We can assume that $\|S\| = 1$. Let π be as in Lemma 1.52. Then, by continuity of π^*, the mass of $\pi^*(S)$ on $\pi^{-1}(B_r)$ is bounded by a constant. It is enough to apply Fubini's theorem in order to estimate the mass of $\pi^*(S) \wedge [\widehat{\Delta}_r]$. $\qquad\square$

Recall the following theorem due to Sibony-Wong [SW].

Theorem 1.54. *Let $m > 0$ be a positive constant. Let $\mathscr{F}' \subset \mathscr{F}$ be such that $\mathscr{L}(\mathscr{F}') \geq m$ and let Σ denote the intersection of the family $\mathscr{F}'$ with B_r. Then any holomorphic function h on a neighbourhood of Σ can be extended to a holomorphic function on $B_{\lambda r}$ where $\lambda > 0$ is a constant depending on m but independent of $\mathscr{F}'$ and r. Moreover, we have*

$$\sup_{B_{\lambda r}} |h| \leq \sup_{\Sigma} |h|.$$

We will use the following version of a lemma due to Briend-Duval [BD2]. Their proof uses the theory of moduli of annuli.

Lemma 1.55. *Let $g : \Delta_r \to \mathbb{P}^k$ be a holomorphic map from a disc of center 0 and of radius r in $\mathbb{C}$. Assume that $\mathrm{area}(g(\Delta_r))$ counted with multiplicity, is smaller than $1/2$. Then for any $\varepsilon > 0$ there is a constant $\lambda > 0$ independent of g, r such that the diameter of $g(\Delta_{\lambda r})$ is smaller than $\varepsilon \sqrt{\mathrm{area}(g(\Delta_r))}$.*

Proof. Observe that the lemma is an easy consequence of the Cauchy formula if g has values in a compact set of $\mathbb{C}^k \subset \mathbb{P}^k$. In order to reduce the problem to this case, it is enough to prove that given an $\varepsilon_0 > 0$, there is a constant $\lambda_0 > 0$ such that $\mathrm{diam}(g(\Delta_{\lambda_0 r})) \leq \varepsilon_0$. For ε_0 small enough, we can apply the above case to g restricted to $\Delta_{\lambda_0 r}$.

By hypothesis, the graphs Γ_g of g in $\Delta_r \times \mathbb{P}^k$ have bounded area. So, according to Bishop's theorem [BS], these graphs form a relatively compact family of analytic sets, that is, the limits of these graphs in the Hausdorff sense, are analytic sets. Since $\mathrm{area}(g(\Delta_r))$ is bounded by $1/2$, the limits have no compact components. So, they are also graphs and the family of the maps g is compact. We deduce that $\mathrm{diam}(g(\Delta_{\lambda_0 r})) \leq \varepsilon_0$ for λ_0 small enough. $\qquad\square$

Sketch of the proof of Proposition 1.51. The last assertion in the proposition is deduced from the first one and Proposition 1.46 applied to a generic point in B. We obtain that $\| \ ' - \ \| \leq 2\sqrt{\nu}$ for every ν strictly larger than $\nu(R, a)$ which implies the result.

For the first assertion, the idea is to construct inverse branches for many discs passing through a and then to apply Theorem 1.54 in order to construct inverse branches on balls. We can assume that ν is smaller than 1. Choose constants $\delta > 0$, $\varepsilon > 0$ small enough and then a constant $\kappa > 0$ large enough; all independent of n. Fix now the integer n. Recall that $\|(f^n)_*(\omega_{\mathrm{FS}})\| = d^{(k-1)n}$. By Lemmas 1.52 and 1.53, there is a family $\mathscr{F}' \subset \mathscr{F}$ and a constant $r > 0$ such that $\mathscr{L}(\mathscr{F}') > 1 - \delta$ and for any Δ in $\mathscr{F}'$, the mass of $R \wedge [\Delta_{\kappa^2 r}]$ is smaller than ν and the mass of $(f^n)_*(\omega_{\mathrm{FS}}) \wedge [\Delta_{\kappa r}]$ is smaller than $Ad^{(k-1)n}$ with $A > 0$.

Claim. *For each Δ in $\mathscr{F}'$, f^n admits at least $(1 - 2\nu)d^{kn}$ inverse branches $g_i : \Delta_{\kappa^2 r} \to V_i$ with $\mathrm{area}(V_i) \leq A\nu^{-1}d^{-n}$. The inverse branches g_i can be extended to a neighbourhood of $\Delta_{\kappa^2 r}$.*

Assuming the claim, we complete the proof of the proposition. Let $a_1, \dots, a_l$ be the points in $f^{-n}(a)$, with $l \leq d^{kn}$, and $\mathscr{F}'_s \subset \mathscr{F}'$ the family of Δ's such that one of the previous inverse branches $g_i : \Delta_{\kappa^2 r} \to V_i$ passes through a_s, that is, V_i contains a_s. The above claim implies that $\sum \mathscr{L}(\mathscr{F}'_s) \geq (1 - \delta)(1 - 2\nu)d^{kn}$. There are at most d^{kn} terms in this sum. We only consider the family $\mathscr{S}$ of the indices s such that $\mathscr{L}(\mathscr{F}'_s) \geq 1 - 3\sqrt{\nu}$. Since $\mathscr{L}(\mathscr{F}'_s) \leq 1$ for every s, we have

$$\#\mathscr{S} + (d^{kn} - \#\mathscr{S})(1 - 3\sqrt{\nu}) \geq \sum \mathscr{L}(\mathscr{F}'_s) \geq (1 - \delta)(1 - 2\nu)d^{kn}.$$

Therefore, since δ is small, we have $\#\mathscr{S} \geq (1 - \sqrt{\nu})d^{kn}$. For any index $s \in \mathscr{S}$ and for Δ in $\mathscr{F}'_s$, by Lemma 1.55, the corresponding inverse branch on $\Delta_{\kappa r}$, which

passes through a_s, has diameter $\leq \varepsilon d^{-n/2}$. By Theorem 1.54, f^n admits an inverse branch defined on the ball B_r and passing through a_s, with diameter $\leq d^{-n/2}$. This implies the result.

Proof of the claim. Let v_l denote the mass of $R_l \wedge [\Delta_{\kappa^2 r}]$. Then, $\sum v_l$ is the mass of $R \wedge [\Delta_{\kappa^2 r}]$. Recall that this mass is smaller than v. By definition, $v_l d^{kl}$ is the number of points in $f^l(Y) \cap \Delta_{\kappa^2 r}$, counted with multiplicity. We only have to consider the case $v < 1$. So, we have $v_0 = 0$ and $\Delta_{\kappa^2 r}$ does not intersect Y, the critical values of f. It follows that $\Delta_{\kappa^2 r}$ admits d^k inverse branches for f. By definition of v_1, there are at most $v_1 d^k$ such inverse branches which intersect Y, i.e. the images intersect Y. So, $(1 - v_1)d^k$ of them do not meet Y and the image of such a branch admits d^k inverse branches for f. We conclude that $\Delta_{\kappa^2 r}$ admits at least $(1 - v_1)d^{2k}$ inverse branches for f^2. By induction, we construct for f^n at least $(1 - v_1 - \cdots - v_{n-1})d^{kn}$ inverse branches on $\Delta_{\kappa^2 r}$.

Now, observe that the mass of $(f^n)_*(\omega_{FS}) \wedge [\Delta_{\kappa r}]$ is exactly the area of $f^{-n}(\Delta_{\kappa r})$. We know that it is smaller than $A d^{(k-1)n}$. It is not difficult to see that $\Delta_{\kappa^2 r}$ has at most $v d^{kn}$ inverse branches with area $\geq A v^{-1} d^{-n}$. This completes the proof. $\square$

End of the proof of Theorem 1.45. Let a be a point out of the exceptional set $\mathscr{E}$ defined in Theorem 1.47 for $X = \mathbb{P}^k$. Fix $\varepsilon > 0$ and a constant $\alpha > 0$ small enough. If $'$ is a limit value of $d^{-kn}(f^n)^*(\delta_a)$, it is enough to show that $\| ' - \| \leq 2\alpha + 2\varepsilon$. Consider $Z := \{v(R, z) > \varepsilon\}$ and τ as in Proposition 1.50 for $X = \mathbb{P}^k$. We have $\tau(a) = 0$. So, for r large enough we have $\tau_r(a) \leq \alpha$. Consider all the negative orbits $\mathscr{O}_j$ of order $r_j \leq r$

$$\mathscr{O}_j = \left\{ a^{(j)}_{-r_j}, \ldots, a^{(j)}_{-1}, a^{(j)}_0 \right\}$$

with $f(a^{(j)}_{-i-1}) = a^{(j)}_{-i}$ and $a^{(j)}_0 = a$ such that $a^{(j)}_{-r_j} \notin Z$ and $a^{(j)}_{-i} \in Z$ for $i \neq r_j$. Each orbit is repeated according to its multiplicity. Let S_r denote the family of points $b \in f^{-r}(a)$ such that $f^i(b) \in Z$ for $0 \leq i \leq r$. Then $f^{-r}(a) \setminus S_r$ consists of the preimages of the points $a^{(j)}_{-r_j}$. So, by definition of τ_r, we have

$$d^{-kr} \# S_r = \tau_r(a) \leq \alpha$$

and

$$d^{-kr} \sum_j d^{k(r-r_j)} = d^{-kr} \#(f^{-r}(a) \setminus S_r) = 1 - \tau_r(a) \geq 1 - \alpha.$$

We have for $n \geq r$

$$d^{-kn}(f^n)^*(\delta_a) = d^{-kn} \sum_{b \in S_r} (f^{(n-r)})^*(\delta_b) + d^{-kn} \sum_j (f^{(n-r_j)})^* \left(\delta_{a^{(j)}_{-r_j}} \right).$$

Since $d^{-kn}(f^n)^*$ preserves the mass of any measure, the first term in the last sum is of mass $d^{-kr} \# S_r = \tau_r(a) \leq \alpha$ and the second term is of mass $\geq 1 - \alpha$. We apply Proposition 1.51 to the Dirac masses at $a^{(j)}_{-r_j}$. We deduce that if $'$ is a limit value of $d^{-kn}(f^n)^*(\delta_a)$ then

$$\| ' - \| \leq 2\alpha + 2\varepsilon.$$

This completes the proof of the theorem. $\square$

We have the following more general result. When X is not normal, one has an analogous result for the lift of g to the normalization of X.

Theorem 1.56. *Let X be an irreducible analytic set of dimension p, invariant under f. Let g denote the restriction of f to X and $\mathscr{E}_X$ the exceptional set of g. Assume that X is a normal analytic space. Then $d^{-pn}(g^n)^*(\delta_a)$ converge to $\mu_X := (\deg X)^{-1} T^p \wedge [X]$ if and only if a is out of $\mathscr{E}_X$. Moreover, the convex set of probability measures on X which are totally invariant under g, is of finite dimension.*

Proof. The proof of the first assertion follows the same lines as in Theorem 1.45. We use the fact that g is the restriction of a holomorphic map in $\mathbb{P}^k$ in order to define the ramification current R. The assumption that X is normal allows to define $d^{-pn}(g^n)^*(\delta_a)$. We prove the second assertion. Observe that an analytic set, totally invariant by g^n, is not necessarily totally invariant by g, but it is a union of components of such sets, see Theorem 1.47. Therefore, we can replace g with an iterate g^n in order to assume that g fixes all the components of all the totally invariant analytic sets. Let μ' be an extremal element in the convex set of totally invariant probability measures and X' the smallest totally invariant analytic set such that $\mu'(X') = 1$. The first assertion applied to X' implies that $\mu' = \mu_{X'}$. Hence, the set of such μ' is finite. We use a normalization of X' if necessary. $\qquad\square$

The following result due to Briend-Duval [BD1], shows that repelling periodic points are equidistributed on the support of the Green measure.

Theorem 1.57. *Let P_n denote the set of repelling periodic points of period n on the support of μ. Then the sequence of measures*

$$\mu_n := d^{-kn} \sum_{a \in P_n} \delta_a$$

converges to μ.

Proof. By Proposition 1.3, the number of periodic points of period n of f, counted with multiplicity, is equal to $(d^n - 1)^{-1}(d^{k(n+1)} - 1)$. Therefore, any limit value μ' of μ_n is of mass ≤ 1. Fix a small constant $\varepsilon > 0$. It is enough to check that for μ-almost every point $a \in \mathbb{P}^k$, there is a ball B centered at a, arbitrarily small, such that $\#P_n \cap B \geq (1 - \varepsilon)d^{kn} \mu(B)$ for large n. We will use in particular a trick due to X. Buff, which simplifies the original proof.

Since μ is PC, it has no mass on analytic sets. So, it has no mass on the orbit $\mathscr{O}_Y$ of Y, the set of critical values of f. Fix a point a on the support of μ and out of $\mathscr{O}_Y$. We have $\nu(R, a) = 0$. By Proposition 1.51, there is a ball B of center a, with sufficiently small radius, which admits $(1 - \varepsilon^2)d^{kn}$ inverse branches of diameter $\leq d^{-n/2}$ for f^n when n is large enough. Choose a finite family of such balls B_i of center b_i with $1 \leq i \leq m$ such that $\mu(B_1 \cup \ldots \cup B_m) > 1 - \varepsilon^2 \mu(B)$ and each B_i admits $(1 - \varepsilon^2 \mu(B))d^{kn}$ inverse branches of diameter $\leq d^{-n/2}$ for f^n when n is large enough. Choose balls $B_i' \Subset B_i$ such that $\mu(B_1' \cup \ldots \cup B_m') > 1 - \varepsilon^2 \mu(B)$.

Fix a constant N large enough. Since $d^{-kn}(f^n)^*(\delta_a)$ converge to μ, there are at least $(1 - 2\varepsilon^2)d^{kN}$ inverse branches for f^N whose image intersects $\cup B_j'$ and

then with image contained in one of the B_j. In the same way, we show that for n large enough, each B_j admits $(1-2\varepsilon^2)\ (B)d^{k(n-N)}$ inverse branches for f^{n-N} with images in B. Therefore, B admits at least $(1-2\varepsilon^2)^2\ (B)d^{kn}$ inverse branches $g_i : B \to U_i$ for f^n with image $U_i \Subset B$. Observe that every holomorphic map $g : B \to U \Subset B$ contracts the Kobayashi metric and then admits an attractive fixed point z. Moreover, g^l converges uniformly to z and $\cap_l g^l(\overline{B}) = \{z\}$. Therefore, each g_i admits a fixed attractive point a_i. This point is fixed and repelling for f^n. They are different since the U_i are disjoint. Finally, since   is totally invariant, its support is also totally invariant under f. Hence, a_i, which is equal to $\cap_l g_i^l(\mathrm{supp}(\ )\cap\overline{B})$, is necessarily in supp( ). We deduce that

$$\#P_n \cap B \geq (1-2\varepsilon^2)^2\ (B)d^{kn} \geq (1-\varepsilon)d^{kn}\ (B).$$

This completes the proof.   □

Note that the periodic points a_i, constructed above, satisfy $\|(Df^n)^{-1}(a_i)\| \lesssim d^{-n/2}$. Note also that in the previous theorem, one can replace P_n with the set of all periodic points counting with multiplicity or not.

Exercise 1.58. Let f be an endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$ as above. Let K be a compact set such that $f^{-1}(K) \subset K$. Show that either K contains $\mathscr{J}_k$, the Julia set of order k, or K is contained in the exceptional set $\mathscr{E}$. Prove that $a \notin \mathscr{E}$ if and only if $\cup f^{-n}(a)$ is Zariski dense.

Exercise 1.59. Let f be as above and U an open set which intersects the Julia set $\mathscr{J}_k$. Show that $\cup_{n\geq 0} f^n(U)$ is a Zariski dense open set of $\mathbb{P}^k$. Deduce that f is topologically transitive on $\mathscr{J}_k$, that is, for any given non-empty open sets V, W on $\mathscr{J}_k$, there is an integer $n \geq 0$ such that $f^n(V) \cap W \neq \varnothing$. If $\mathscr{E} = \varnothing$, show that $f^n(U) = \mathbb{P}^k$ for n large enough. If $\mathscr{E} \cap \mathscr{J}_k = \varnothing$, show that $f^n(V) = \mathscr{J}_k$ for n large enough.

Exercise 1.60. Assume that p is a repelling fixed point in $\mathscr{J}_k$. If g is another endomorphism close enough to f in $\mathscr{H}_d(\mathbb{P}^k)$ such that $g(p) = p$, show that p belongs also to the Julia set of order k of g. Hint: use that $g \mapsto\ {}_g$ is continuous.

Exercise 1.61. Using Example 1.10, construct a map f in $\mathscr{H}_d(\mathbb{P}^k)$, $d \geq 2$, such that for n large enough, every fiber of f^n contains more than $d^{(k-1/2)n}$ points. Deduce that there is Zariski dense open set in $\mathscr{H}_d(\mathbb{P}^k)$ such that if f is in that Zariski open set, its exceptional set is empty.

Exercise 1.62. Let ε be a fixed constant such that $0 < \varepsilon < 1$. Let P'_n the set of repelling periodic points a of prime period n on the support of   such that all the eigenvalues of Df^n at a are of modulus $\geq (d-\varepsilon)^{n/2}$. Show that $d^{-kn}\sum_{a\in P'_n}\delta_a$ converges to .

Exercise 1.63. Let g be as in Theorem 1.56. Show that repelling periodic points on supp$(\ _X)$ are equidistributed with respect to $_X$. In particular, they are Zariski dense.

1.5 Equidistribution of Varieties

In this paragraph, we consider the inverse images by f^n of varieties in $\mathbb{P}^k$. The geometrical method in the last paragraph is quite difficult to apply here. Indeed, the inverse image of a generic variety of codimension $p < k$ is irreducible of degree $O(d^{pn})$. The pluripotential method that we introduce here is probably the right method for the equidistribution problem. Moreover, it should give some precise estimates on the convergence, see Remark 1.71.

The following result, due to the authors, gives a satisfactory solution in the case of hypersurfaces. It was proved for Zariski generic maps by Fornæss-Sibony in [FS3, S3] and for maps in dimension 2 by Favre-Jonsson in [FJ]. More precise results are given in [DS9] and in [FJ, FJ1] when $k = 2$. The proof requires some self-intersection estimates for currents, due to Demailly-Méo.

Theorem 1.64. *Let f be an endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$. Let $\mathscr{E}_m$ denote the union of the totally invariant proper analytic sets in $\mathbb{P}^k$ which are minimal, i.e. do not contain smaller ones. Let S be a positive closed $(1,1)$-current of mass 1 on $\mathbb{P}^k$ whose local potentials are not identically $-\infty$ on any component of $\mathscr{E}_m$. Then, $d^{-n}(f^n)^*(S)$ converge weakly to the Green $(1,1)$-current T of f.*

The following corollary gives a solution to the equidistribution problem for hypersurfaces: the exceptional hypersurfaces belong to a proper analytic set in the parameter space of hypersurfaces of a given degree.

Corollary 1.65. *Let f, T and $\mathscr{E}_m$ be as above. If H is a hypersurface of degree s in $\mathbb{P}^k$, which does not contain any component of $\mathscr{E}_m$, then $s^{-1}d^{-n}(f^n)^*[H]$ converge to T in the sense of currents.*

Note that $(f^n)^*[H]$ is the current of integration on $f^{-n}(H)$ where the components of $f^{-n}(H)$ are counted with multiplicity.

Sketch of the proof of Theorem 1.64. We can write $S = T + dd^c u$ where u is a p.s.h. function modulo T, that is, the difference of quasi-potentials of S and of T. Subtracting from u a constant allows to assume that $\langle \ , u \rangle = 0$. We call u *the dynamical quasi-potential* of S. Since T has continuous quasi-potentials, u satisfies analogous properties that quasi-p.s.h. functions do. We are mostly concerned with the singularities of u.

The total invariance of T and   implies that the dynamical quasi-potential of $d^{-n}(f^n)^*(S)$ is equal to $u_n := d^{-n}u \circ f^n$. We have to show that this sequence of functions converges to 0 in $L^1(\mathbb{P}^k)$. Since u is bounded from above, we have $\limsup u_n \leq 0$. Assume that u_n do not converge to 0. By Hartogs' lemma, see Proposition A.20, there is a ball B and a constant $\lambda > 0$ such that $u_n \leq -\lambda$ on B for infinitely many indices n. It follows that $u \leq -\lambda d^n$ on $f^n(B)$ for such an index n. On the other hand, the exponential estimate in Theorem A.22 implies that $\|e^{\alpha|u|}\|_{L^1} \leq A$ for some positive constants α and A independent of u. If the multiplicity of f at

every point is $\leq d - 1$, then a version of Lojasiewicz's theorem implies that $f^n(B)$ contains a ball of radius $\simeq e^{-c(d-1)^n}$, $c > 0$. Therefore, we have

$$e^{-2kc(d-1)^n} e^{\lambda d^n \alpha} \lesssim \int_{f^n(B)} e^{\lambda d^n \alpha} \omega_{FS}^k \leq \int_{\mathbb{P}^k} e^{\alpha|u|} \omega_{FS}^k.$$

This contradicts the above exponential estimate.

In general, by Lemma 1.66 below, $f^n(B)$ contains always a ball of radius $\simeq e^{-cd^n}$. So, a slightly stronger version of the above exponential estimate will be enough to get a contradiction. We may improve this exponential estimate: if the Lelong numbers of S are small, we can increase the constant α and get a contradiction; if the Lelong numbers of S_n are small, we replace S with S_n.

The assumption $u_n < -\lambda d^n$ on $f^n(B)$ allows to show that all the limit currents of the sequence $d^{-n}(f^n)^*(S)$ have Lelong numbers larger than some constant $v > 0$. If S' is such a current, there are other currents S'_n such that $S' = d^{-n}(f^n)^*(S'_n)$. Indeed, if S' is the limit of $d^{-n_i}(f^{n_i})^*(S)$ one can take S'_n a limit value of $d^{-n_i+n}(f^{n_i-n})^*(S)$.

Let a be a point such that $v(S'_n, a) \geq v$. The assumption on the potentials of S allows to prove by induction on the dimension of the totally invariant analytic sets that u_n converge to 0 on the maximal totally invariant set $\mathscr{E}$. So, a is out of $\mathscr{E}$. Lemma 1.49 allows to construct many distinct points in $f^{-n}(a)$. The identity $S' = d^{-n}(f^n)^*(S'_n)$ implies an estimate from below of the Lelong numbers of S' on $f^{-n}(a)$. This holds for every n. Finally, this permits to construct analytic sets of large degrees on which we have estimates on the Lelong numbers of S'. Therefore, S' has a too large self-intersection. This contradicts an inequality of Demailly-Méo [DE,ME3] and completes the proof. Note that the proof of Demailly-Méo inequality uses Hörmander's L^2 estimates for the $\bar{\partial}$-equation. $\square$

The following lemma is proved in [DS9]. It also holds for meromorphic maps. Some earlier versions were given in [FS3] and in terms of Lebesgue measure in [FJ,G2].

Lemma 1.66. *There is a constant $c > 0$ such that if B is a ball of radius r, $0 < r < 1$, in $\mathbb{P}^k$, then $f^n(B)$ contains a ball B_n of radius $\exp(-cr^{-2k}d^n)$ for any $n \geq 0$.*

The ball B_n is centered at $f^n(a_n)$ for some point $a_n \in B$ which is not necessarily the center of B. The key point in the proof of the lemma is to find a point a_n with an estimate from below on the Jacobian of f^n at a_n. If u is a quasi-potential of the current of integration on the critical set, the logarithm of this Jacobian is essentially the value of $u + u \circ f + \cdots + u \circ f^{n-1}$ at a_n. So, in order to prove the existence of a point a_n with a good estimate, it is enough to bound the L^1 norm of the last function. One easily obtains the result using the operator $f^* : \mathrm{DSH}(\mathbb{P}^k) \to \mathrm{DSH}(\mathbb{P}^k)$ and its iterates, as it is done for f_* in Proposition 1.34.

Remark 1.67. Let $\mathscr{C}$ denote the convex compact set of totally invariant positive closed $(1,1)$-currents of mass 1 on $\mathbb{P}^k$, i.e. currents S such that $f^*(S) = dS$. Define an operator $\vee$ on $\mathscr{C}$. If S_1, S_2 are elements of $\mathscr{C}$ and u_1, u_2 their dynamical quasi-potentials, then $u_i \leq 0$. Since $\langle \ , u_i \rangle = 0$ and u_i are upper semi-continuous,

we deduce that $u_i = 0$ on supp( ). Define $S_1 \vee S_2 := T + dd^c \max(u_1, u_2)$. It is easy
to check that $S_1 \vee S_2$ is an element of $\mathscr{C}$. An element S is said to be *minimal* if
$S = S_1 \vee S_2$ implies $S_1 = S_2 = S$. It is clear that T is not minimal if $\mathscr{C}$ contains other
currents. In fact, for S in $\mathscr{C}$, we have $T \vee S = T$. A current of integration on a totally
invariant hypersurface is a minimal element. It is likely that $\mathscr{C}$ is generated by a
finite number of currents, the operation $\vee$, convex hulls and limits.

Example 1.68. If f is the map given in Example 1.11, the exceptional set $\mathscr{E}_m$ is the
union of the $k+1$ attractive fixed points

$$[0 : \cdots : 0 : 1 : 0 : \cdots : 0].$$

The convergence of $s^{-1} d^{-n} (f^n)^* [H]$ towards T holds for hypersurfaces H of degree
s which do not contain these points. If $\pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k$ is the canonical pro-
jection, the Green $(1,1)$-current T of f is given by $\pi^*(T) = dd^c(\max_i \log|z_i|)$, or
equivalently $T = \omega_{FS} + dd^c v$ where

$$v[z_0 : \cdots : z_k] := \max_{0 \leq i \leq k} \log|z_i| - \frac{1}{2} \log(|z_0|^2 + \cdots + |z_k|^2).$$

The currents T_i of integration on $\{z_i = 0\}$ belong to $\mathscr{C}$ and $T_i = T + dd^c u_i$ with
$u_i := \log|z_i| - \max_j \log|z_j|$. These currents are minimal. If $\alpha_0, \ldots, \alpha_k$ are positive
real numbers such that $\alpha := 1 - \sum \alpha_i$ is positive, then $S := \alpha T + \sum \alpha_i T_i$ is an element
of $\mathscr{C}$. We have $S = T + dd^c u$ with $u := \sum \alpha_i u_i$. The current S is minimal if and only
if $\alpha = 0$. One can obtain other elements of $\mathscr{C}$ using the operator $\vee$. We show that $\mathscr{C}$
is infinite dimensional. Define for $A := (\alpha_0, \ldots, \alpha_k)$ with $0 \leq \alpha_i \leq 1$ and $\sum \alpha_i = 1$
the p.s.h. function v_A by

$$v_A := \sum \alpha_i \log|z_i|.$$

If $\mathscr{A}$ is a family of such $(k+1)$-tuples A, define

$$v_{\mathscr{A}} := \sup_{A \in \mathscr{A}} v_A.$$

Then, we can define a positive closed $(1,1)$-currents $S_{\mathscr{A}}$ on $\mathbb{P}^k$ by $\pi^*(S_{\mathscr{A}}) = dd^c v_{\mathscr{A}}$.
It is clear that $S_{\mathscr{A}}$ belongs to $\mathscr{C}$ and hence $\mathscr{C}$ is of infinite dimension.

The equidistribution problem in higher codimension is much more delicate and
is still open for general maps. We first recall the following lemma.

Lemma 1.69. *For every $\delta > 1$, there is a Zariski dense open set $\mathscr{H}_d^*(\mathbb{P}^k)$ in $\mathscr{H}_d(\mathbb{P}^k)$
and a constant $A > 0$ such that for f in $\mathscr{H}_d^*(\mathbb{P}^k)$, the maximal multiplicity δ_n of f^n
at a point in $\mathbb{P}^k$ is at most equal to $A\delta^n$. In particular, the exceptional set of such a
map f is empty when $\delta < d$.*

Proof. Let X be a component of a totally invariant analytic set E of pure dimension
$p \leq k - 1$. Then, f permutes the components of E. We deduce that X is totally in-
variant under f^n for some $n \geq 1$. Lemma 1.48 implies that the maximal multiplicity

of f^n at a point in X is at least equal to $d^{(k-p)n}$. Therefore, the second assertion in the lemma is a consequence of the first one.

Fix an N large enough such that $\delta^N > 2^k k!$. Let $\mathscr{H}_d^*(\mathbb{P}^k)$ be the set of f such that $\delta_N \leq 2^k k!$. This set is Zariski open in $\mathscr{H}_d(\mathbb{P}^k)$. Since the sequence (δ_n) is sub-multiplicative, i.e. $\delta_{n+m} \leq \delta_n \delta_m$ for $n, m \geq 0$, if f is in $\mathscr{H}_d^*(\mathbb{P}^k)$, we have $\delta_N < \delta^N$, hence $\delta_n \leq A\delta^n$ for A large enough and for all n. It remains to show that $\mathscr{H}_d^*(\mathbb{P}^k)$ is not empty. Choose a rational map $h : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d whose critical points are simple and have disjoint infinite orbits. Observe that the multiplicity of h^N at every point is at most equal to 2. We construct the map f using the method described in Example 1.10. We have $f^N \circ \Pi = \Pi \circ \widehat{f}^N$. Consider a point x in $\mathbb{P}^k$ and a point $\widehat{x}$ in $\Pi^{-1}(x)$. The multiplicity of $\widehat{f}^N$ at $\widehat{x}$ is at most equal to 2^k. It follows that the multiplicity of f^N at x is at most equal to $2^k k!$ since Π has degree $k!$. Therefore, f satisfies the desired inequality. $\square$

We have the following result due to the authors [DS10].

Theorem 1.70. *There is a Zariski dense open set $\mathscr{H}_d^*(\mathbb{P}^k)$ in $\mathscr{H}_d(\mathbb{P}^k)$ such that if f is in $\mathscr{H}_d^*(\mathbb{P}^k)$, then $d^{-pn}(f^n)^*(S) \to T^p$ uniformly on positive closed (p, p)-currents S of mass 1 on $\mathbb{P}^k$. In particular, the Green (p, p)-current T^p is the unique positive closed (p, p)-current of mass 1 which is totally invariant. If V is an analytic set of pure codimension p and of degree s in $\mathbb{P}^k$, then $s^{-1}d^{-pn}(f^n)^*[V]$ converge to T^p in the sense of currents.*

Sketch of proof. The proof uses the super-potentials of currents. In order to simplify the notation, introduce the dynamical super-potential $\mathscr{V}$ of S. Define $\mathscr{V} := \mathscr{U}_S - \mathscr{U}_{T^p} + c$ where $\mathscr{U}_S, \mathscr{U}_{T^p}$ are super-potentials of S, T^p and the constant c is chosen so that $\mathscr{V}(T^{k-p+1}) = 0$. Using a computation as in Theorem 1.32, we obtain that the dynamical super-potential of $d^{-pn}(f^n)^*(S)$ is equal to $d^{-n}\mathscr{V} \circ \Lambda^n$ where $\Lambda : \mathscr{C}_{k-p+1}(\mathbb{P}^k) \to \mathscr{C}_{k-p+1}(\mathbb{P}^k)$ is the operator $d^{-p+1}f_*$. Observe that the dynamical super-potential of T^p is identically 0. In order to prove the convergence $d^{-pn}(f^n)^*(S) \to T^p$, we only have to check that $d^{-n}\mathscr{V}(\Lambda^n(R)) \to 0$ for R smooth in $\mathscr{C}_{k-p+1}(\mathbb{P}^k)$. Since T^p has a continuous super-potential, $\mathscr{V}$ is bounded from above. Therefore, $\limsup d^{-n}\mathscr{V}(\Lambda^n(R)) \leq 0$.

Recall that $\mathscr{U}_S(R) = \mathscr{U}_R(S)$. So, in order to prove that $\liminf d^{-n}\mathscr{V}(\Lambda^n(R)) \geq 0$, it is enough to estimate $\inf_S \mathscr{U}_S(\Lambda^n(R))$, or equivalently, we have to estimate the capacity of $\Lambda^n(R)$ from below. Assume in order to explain the idea that the support of R is contained in a compact set K such that $f(K) \subset K$ and K does not intersect the critical set of f (this is possible only when $p = 1$). We easily obtain that $\|\Lambda^n(R)\|_\infty \lesssim A^n$ for some constant $A > 0$. The estimate in Theorem A.47 implies the result. In the general case, if $\mathscr{H}_d^*(\mathbb{P}^k)$ is chosen as in Lemma 1.69 for δ small enough and if f is in $\mathscr{H}_d^*(\mathbb{P}^k)$, we can prove the estimate $\mathrm{cap}(\Lambda^n(R)) \lesssim d'^n$ for any fixed constant d' such that $1 < d' < d$. This implies the desired convergence of super-potentials. The main technical difficulty is that when R hits the critical set, then $\Lambda(R)$ is not bounded. The estimates requires a smoothing and precise evaluation of the error. $\square$

Remark 1.71. The above estimate on $\mathrm{cap}(\Lambda^n(R))$ can be seen as a version of Lojasiewicz's inequality for currents. It is quite delicate to obtain. We also have an explicit estimate on the speed of convergence. Indeed, we have for an appropriate $d' < d$:

$$\mathrm{dist}_2\big(d^{-pn}(f^n)^*(S), T^p\big) := \sup_{\|\Phi\|_{\mathscr{C}^2} \leq 1} |\langle d^{-pn}(f^n)^*(S) - T^p, \Phi\rangle| \lesssim d'^n d^{-n}.$$

The theory of interpolation between Banach spaces [T1] implies a similar estimate for Φ Hölder continuous.

Exercise 1.72. If $f := [z_0^d : \cdots : z_k^d]$, show that $\{z_1^p = z_2^q\}$, for arbitrary p, q, is invariant under f. Show that a curve invariant under an endomorphism is an image of $\mathbb{P}^1$ or a torus, possibly singular.

Exercise 1.73. Let f be as in Example 1.68. Let S be a (p, p)-current with strictly positive Lelong number at $[1 : 0 : \cdots : 0]$. Show that any limit of $d^{-pn}(f^n)^*(S)$ has a strictly positive Lelong number at $[1 : 0 : \cdots : 0]$ and deduce that $d^{-pn}(f^n)^*(S)$ do not converge to T^p. Note that for f generic, the multiplicity of the set of critical values of f^N at every point is smaller than δ^N.

Exercise 1.74. Let f be as in Theorem 1.70 for $p = k$ and Λ the associated Perron-Frobenius operator. If φ is a $\mathscr{C}^2$ function on $\mathbb{P}^k$, show that

$$\|\Lambda^n(\varphi) - \langle \ , \varphi\rangle\|_\infty \leq cd'^n d^{-n}$$

for some constant $c > 0$. Deduce that $\Lambda^n(\varphi)$ converge uniformly to $\langle \ , \varphi\rangle$. Give an estimate of $\|\Lambda^n(\varphi) - \langle \ , \varphi\rangle\|_\infty$ for φ Hölder continuous.

Exercise 1.75. Let f be an endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$. Assume that V is a totally invariant hypersurface, i.e. $f^{-1}(V) = V$. Let V_i denote the irreducible components of V and h_i minimal homogeneous polynomials such that $V_i = \{h_i = 0\}$. Define $h = \prod h_i$. Show that $h \circ f = ch^d$ where c is a constant. If F is a lift of f to $\mathbb{C}^{k+1}$, prove that $\mathrm{Jac}(F)$ contains $(\prod h_i)^{d-1}$ as a factor. Show that V is contained in the critical set of f and deduce[3] that $\deg V \leq k + 1$. Assume now that V is reducible. Find a totally invariant positive closed $(1, 1)$-current of mass 1 which is not the Green current nor the current associated to an analytic set.

Exercise 1.76. Let u be a p.s.h. function in $\mathbb{C}^k$, such that for $\lambda \in \mathbb{C}^*$, $u(\lambda z) = \log|\lambda| + u(z)$. If $\{u < 0\}$ is bounded in $\mathbb{C}^k$, show that $dd^c u^+$ is a positive closed current on $\mathbb{P}^k$ which is extremal in the cone of positive closed $(1, 1)$-currents[4]. Deduce that the Green $(1, 1)$-current of a polynomial map of $\mathbb{C}^k$ which extends holomorphically to $\mathbb{P}^k$, is extremal.

Exercise 1.77. Let v be a subharmonic function on $\mathbb{C}$. Suppose $v(e^{i\theta}z) = v(z)$ for every $z \in \mathbb{C}$ and for every $\theta \in \mathbb{R}$ such that $e^{i\theta d^n} = 1$ for some integer n. Prove

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[3] It is known that in dimension $k = 2$, V is a union of at most 3 lines, [CL, FS7, SSU].

[4] Unpublished result by Berndtsson-Sibony.

that $v(z) = v(|z|)$ for $z \in \mathbb{C}$. Hint: use the Laplacian of v. Let f be as in Example 1.68, R a current in $\mathbb{C}^{k+1}$, and v a p.s.h. function on $\mathbb{C}^{k+1}$ such that $R = dd^c v$ and $v(F(z)) = dv(z)$, where $F(z) := (z_0^d, \ldots, z_k^d)$ is a lift of f to $\mathbb{C}^{k+1}$. Show that v is invariant under the action of the unit torus $\mathbb{T}^{k+1}$ in $\mathbb{C}^{k+1}$. Determine such functions v. Recall that $\mathbb{T}$ is the unit circle in $\mathbb{C}$ and $\mathbb{T}^{k+1}$ acts on $\mathbb{C}^{k+1}$ by multiplication.

Exercise 1.78. Define the Desboves map f_0 in $\mathscr{M}_4(\mathbb{P}^2)$ as

$$f_0[z_0 : z_1 : z_2] := [z_0(z_1^3 - z_2^3) : z_1(z_2^3 - z_0^3) : z_2(z_0^3 - z_1^3)].$$

Prove that f_0 has 12 indeterminacy points. If σ is a permutation of coordinates, compare $f_0 \circ \sigma$ and $\sigma \circ f_0$. Define

$$\Phi_\lambda(z_0, z_1, z_2) := z_0^3 + z_1^3 + z_2^3 - 3\lambda z_0 z_1 z_2, \quad \lambda \in \mathbb{C}$$

and

$$L[z_0 : z_1 : z_2] := [az_0 : bz_1 : cz_2], \quad a, b, c \in \mathbb{C}.$$

Show that for Zariski generic L, $f_L := f_0 + \Phi_\lambda L$ is in $\mathscr{H}_4(\mathbb{P}^2)$. Show that on the curve $\{\Phi_\lambda = 0\}$ in $\mathbb{P}^2$, f_L coincides with f_0, and that f_0 maps the cubic $\{\Phi_\lambda = 0\}$ onto itself.[5]

Exercise 1.79. Let f be an endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$. Show that there is a finite invariant set E, possibly empty, such that if H is a hypersurface such that $H \cap E = \varnothing$, then $d^{-n} \deg(H)^{-1}(f^n)^*[H]$ converge to the Green $(1,1)$-current T of f.

1.6 Stochastic Properties of the Green Measure

In this paragraph, we are concerned with the stochastic properties of the equilibrium measure   associated to an endomorphism f. If φ is an observable, $(\varphi \circ f^n)_{n \geq 0}$ can be seen as a sequence of dependent random variables. Since the measure is invariant, these variables are identically distributed, i.e. the Borel sets $\{\varphi \circ f^n < t\}$ have the same   measure for any fixed constant t. The idea is to show that the dependence is weak and then to extend classical results in probability theory to our setting. One of the key point is the spectral study of the Perron-Frobenius operator $\Lambda := d^{-k} f_*$. It allows to prove the exponential decay of correlations for d.s.h. and Hölder continuous observables, the central limit theorem, the large deviation theorem, etc. An important point is to use the space of d.s.h. functions as a space of observables. For the reader's convenience, we recall few general facts from ergodic theory and probability theory. We refer to [KH, W] for the general theory.

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[5] This example was considered in [BO]. It gives maps in $\mathscr{H}_4(\mathbb{P}^2)$ which preserves a cubic. The cubic is singular if $\lambda = 1$, non singular if $\lambda \neq 1$. In higher dimension, Beauville proved that a smooth hypersurface of $\mathbb{P}^k$, $k \geq 3$, of degree > 1 does not have an endomorphism with $d_t > 1$, unless the degree is 2, $k = 3$ and the hypersurface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ [BV].

Consider a dynamical system associated to a map $g : X \to X$ which is measurable with respect to a σ-algebra $\mathscr{F}$ on X. The direct image of a probability measure ν by g is the probability measure $g_*(\nu)$ defined by

$$g_*(\nu)(A) := \nu(g^{-1}(A))$$

for every measurable set A. Equivalently, for any positive measurable function φ, we have

$$\langle g_*(\nu), \varphi \rangle := \langle \nu, \varphi \circ g \rangle.$$

The measure ν is *invariant* if $g_*(\nu) = \nu$. When X is a compact metric space and g is continuous, the set $\mathscr{M}(g)$ of invariant probability measures is convex, compact and non-empty: for any sequence of probability measures ν_N, the cluster points of

$$\frac{1}{N} \sum_{n=0}^{N-1} (g^n)_*(\nu_N)$$

are invariant probability measures.

A measurable set A is *totally invariant* if $\nu(A \setminus g^{-1}(A)) = \nu(g^{-1}(A) \setminus A) = 0$. An invariant probability measure ν is *ergodic* if any totally invariant set is of zero or full ν-measure. It is easy to show that ν is ergodic if and only if when $\varphi \circ g = \varphi$, for $\varphi \in L^1(\nu)$, then φ is constant. Here, we can replace $L^1(\nu)$ with $L^p(\nu)$ with $1 \le p \le +\infty$. The ergodicity of ν is also equivalent to the fact that it is extremal in $\mathscr{M}(g)$. We recall Birkhoff's ergodic theorem, which is the analogue of the law of large numbers for independent random variables [W].

Theorem 1.80 (Birkhoff). *Let $g : X \to X$ be a measurable map as above. Assume that ν is an ergodic invariant probability measure. Let φ be a function in $L^1(\nu)$. Then*

$$\frac{1}{N} \sum_{n=0}^{N-1} \varphi(g^n(x)) \to \langle \nu, \varphi \rangle$$

almost everywhere with respect to ν.

When X is a compact metric space, we can apply Birkhoff's theorem to continuous functions φ and deduce that for ν almost every x

$$\frac{1}{N} \sum_{n=0}^{N-1} \delta_{g^n(x)} \to \nu,$$

where δ_x denotes the Dirac mass at x. The sum

$$S_N(\varphi) := \sum_{n=0}^{N-1} \varphi \circ g^n$$

is called *Birkhoff's sum*. So, Birkhoff's theorem describes the behavior of $\frac{1}{N} S_N(\varphi)$ for an observable φ. We will be concerned with the precise behavior of $S_N(\varphi)$ for various classes of functions φ.

A notion stronger than ergodicity is the notion of mixing. An invariant probability measure v is *mixing* if for every measurable sets A, B

$$\lim_{n\to\infty} v(g^{-n}(A)\cap B) = v(A)v(B).$$

Clearly, mixing implies ergodicity. It is not difficult to see that v is mixing if and only if for any test functions φ, ψ in $L^\infty(v)$ or in $L^2(v)$, we have

$$\lim_{n\to\infty}\langle v, (\varphi\circ g^n)\psi\rangle = \langle v, \varphi\rangle\langle v, \psi\rangle.$$

The quantity

$$I_n(\varphi, \psi) := |\langle v, (\varphi\circ g^n)\psi\rangle - \langle v, \varphi\rangle\langle v, \psi\rangle|$$

is called *the correlation at time n* of φ and ψ. So, mixing is equivalent to the convergence of $I_n(\varphi, \psi)$ to 0. We say that v is *K-mixing* if for every $\psi \in L^2(v)$

$$\sup_{\|\varphi\|_{L^2(v)}\leq 1} I_n(\varphi, \psi) \to 0.$$

Note that K-mixing is equivalent to the fact that the σ-algebra $\mathscr{F}_\infty := \cap g^{-n}(\mathscr{F})$ contains only sets of zero and full measure. This is the strongest form of mixing for observables in $L^2(v)$. It is however of interest to get a quantitative information on the mixing speed for more regular observables like smooth or Hölder continuous functions.

Consider now an endomorphism f of algebraic degree $d \geq 2$ of $\mathbb{P}^k$ as above and its equilibrium measure . We know that is totally invariant: $f^*( ) = d^k$. If φ is a continuous function, then

$$\langle \ , \varphi\circ f\rangle = \langle d^{-k}f^*(\ ), \varphi\circ f\rangle = \langle \ , d^{-k}f_*(\varphi\circ f)\rangle = \langle \ , \varphi\rangle.$$

We have used the obvious fact that $f_*(\varphi\circ f) = d^k\varphi$. So, is invariant. We have the following proposition.

Proposition 1.81. *The Perron-Frobenius operator $\Lambda := d^{-k}f_*$ has a continuous extension of norm 1 to $L^2( )$. Moreover, the adjoint of Λ satisfies ${}^t\Lambda(\varphi) = \varphi\circ f$ and $\Lambda\circ{}^t\Lambda = \mathrm{id}$. Let $L_0^2( )$ denote the hyperplane of $L^2( )$ defined by $\langle \ , \varphi\rangle = 0$. Then, the spectral radius of Λ on $L_0^2( )$ is also equal to 1.*

Proof. Schwarz's inequality implies that

$$|f_*(\varphi)|^2 \leq d^k f_*(|\varphi|^2).$$

Using the total invariance of , we get

$$\langle \ , |\Lambda(\varphi)|^2\rangle \leq \langle \ , \Lambda(|\varphi|^2)\rangle = \langle \ , |\varphi|^2\rangle.$$

Therefore, Λ has a continuous extension to $L^2( )$, with norm ≤ 1. Since $\Lambda(1) = 1$, the norm of this operator is equal to 1. The properties on the adjoint of Λ are easily deduced from the total invariance of .

Let φ be a function in $L_0^2(\ )$ of norm 1. Then, $\varphi \circ f^n$ is also in $L^2(\ )$ and of norm 1. Moreover, $\Lambda^n(\varphi \circ f^n)$, which is equal to φ, is of norm 1. So, the spectral radius of Λ on $L_0^2(\ )$ is also equal to 1. $\square$

Mixing for the measure   was proved in [FS1]. We give in this paragraph two proofs of K-mixing. The first one is from [DS1] and does not use that   is moderate.

Theorem 1.82. *Let f be an endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$. Then, its Green measure   is K-mixing.*

Proof. Let $c_\psi := \langle\ , \psi\rangle$. Since   is totally invariant, the correlations between two observables φ and ψ satisfy

$$I_n(\varphi, \psi) = |\langle\ , (\varphi \circ f^n)\psi\rangle - \langle\ , \varphi\rangle\langle\ , \psi\rangle|$$
$$= |\langle\ , \varphi\Lambda^n(\psi)\rangle - c_\psi\langle\ , \varphi\rangle|$$
$$= |\langle\ , \varphi(\Lambda^n(\psi) - c_\psi)\rangle|.$$

Hence,

$$\sup_{\|\varphi\|_{L^2(\ )} \leq 1} I_n(\varphi, \psi) \leq \|\Lambda^n(\psi) - c_\psi\|_{L^2(\ )}.$$

Since $\|\Lambda\|_{L^2(\ )} \leq 1$, in order to show that $\|\Lambda^n(\psi) - c_\psi\|_{L^2(\ )} \to 0$ for every $\psi \in L^2(\ )$, it is enough to show that $\|\Lambda^n(\psi) - c_\psi\|_{L^2(\ )} \to 0$ for a dense family of functions $\psi \in L^2(\ )$. So, we can assume that ψ is a d.s.h. function such that $|\psi| \leq 1$. We have $|c_\psi| \leq 1$ and $\|\Lambda^n(\psi) - c_\psi\|_{L^\infty(\ )} \leq 2$. Since   is PB, we deduce from Theorem 1.35 and Cauchy-Schwarz's inequality that

$$\|\Lambda^n(\psi) - c_\psi\|_{L^2(\ )} \lesssim \|\Lambda^n(\psi) - c_\psi\|_{L^1(\ )}^{1/2} \lesssim \|\Lambda^n(\psi) - c_\psi\|_{\mathrm{DSH}}^{1/2} \lesssim d^{-n/2}.$$

This completes the proof. For the last argument, we can also use continuous test functions ψ and apply Proposition 1.46. We then obtain that $\Lambda^n(\psi) - c_\psi$ converges to 0 pointwise out of a pluripolar set. Lebesgue's convergence theorem and the fact that   has no mass on pluripolar sets imply the result. $\square$

In what follows, we show that the equilibrium measure   satisfies remarkable stochastic properties which are quite hard to obtain in the setting of real dynamical systems. We will see the effectiveness of the pluripotential methods which replace the delicate estimates, used in some real dynamical systems. The following result was recently obtained by Nguyen and the authors [DNS]. It shows that the equilibrium measure is exponentially mixing and generalizes earlier results of [DS1, DS6, FS1]. Note that d.s.h. observables may be everywhere discontinuous.

Theorem 1.83. *Let f be a holomorphic endomorphism of algebraic degree $d \geq 2$ on $\mathbb{P}^k$. Let   be the Green measure of f. Then for every $1 < p \leq +\infty$ there is a constant $c > 0$ such that*

$$|\langle\ , (\varphi \circ f^n)\psi\rangle - \langle\ , \varphi\rangle\langle\ , \psi\rangle| \leq cd^{-n}\|\varphi\|_{L^p(\ )}\|\psi\|_{\mathrm{DSH}}$$

*for $n \geq 0$, φ in $L^p(\ )$ and ψ d.s.h. Moreover, for $0 \leq v \leq 2$ there is a constant $c > 0$
such that*

$$|\langle\ ,(\varphi \circ f^n)\psi\rangle - \langle\ ,\varphi\rangle\langle\ ,\psi\rangle| \leq cd^{-nv/2}\|\varphi\|_{L^p(\ )}\|\psi\|_{\mathscr{C}^v}$$

for $n \geq 0$, φ in $L^p(\ )$ and ψ of class $\mathscr{C}^v$.

Proof. We prove the first assertion. Observe that the correlations

$$I_n(\varphi, \psi) := |\langle\ ,(\varphi \circ f^n)\psi\rangle - \langle\ ,\varphi\rangle\langle\ ,\psi\rangle|$$

vanish if ψ is constant. Therefore, we can assume that $\langle\ ,\psi\rangle = 0$. In which case,
we have

$$I_n(\varphi, \psi) = |\langle\ ,\varphi\Lambda^n(\psi)\rangle|,$$

where Λ denotes the Perron-Frobenius operator associated to f.

We can also assume that $\|\psi\|_{\text{DSH}} \leq 1$. Corollary 1.41 implies that for $1 \leq q < \infty$,

$$\|\Lambda^n(\psi)\|_{L^q(\ )} \leq cqd^{-n}$$

where $c > 0$ is a constant independent of n, q and ψ. Now, if q is chosen so that
$p^{-1} + q^{-1} = 1$, we obtain using Hölder's inequality that

$$I_n(\varphi, \psi) \leq \|\varphi\|_{L^p(\ )}\|\Lambda^n(\psi)\|_{L^q(\ )} \leq cq\|\varphi\|_{L^p(\ )}d^{-n}.$$

This completes the proof of the first assertion. The second assertion is proved in the
same way using Corollary 1.42. □

Observe that the above estimates imply that for ψ smooth

$$\lim_{n\to\infty} \sup_{\|\varphi\|_{L^2(\ )} \leq 1} I_n(\varphi, \psi) = 0.$$

Since smooth functions are dense in $L^2(\ )$, the convergence holds for every ψ in
$L^2(\ )$ and gives another proof of the K-mixing. The following result [DNS] gives
estimates for the exponential mixing of any order. It can be extended to Hölder
continuous observables using the second assertion in Theorem 1.83.

Theorem 1.84. *Let f, d, be as in Theorem 1.83 and $r \geq 1$ an integer. Then there
is a constant $c > 0$ such that*

$$\left|\langle\ ,\psi_0(\psi_1 \circ f^{n_1})\ldots(\psi_r \circ f^{n_r})\rangle - \prod_{i=0}^{r}\langle\ ,\psi_i\rangle\right| \leq cd^{-n}\prod_{i=0}^{r}\|\psi_i\|_{\text{DSH}}$$

for $0 = n_0 \leq n_1 \leq \cdots \leq n_r$, $n := \min_{0 \leq i < r}(n_{i+1} - n_i)$ and ψ_i d.s.h.

Proof. The proof is by induction on r. The case $r = 1$ is a consequence of Theorem
1.83. Suppose the result is true for $r - 1$, we have to check it for r. Without loss

of generality, assume that $\|\psi_i\|_{\mathrm{DSH}} \leq 1$. This implies that $m := \langle\ ,\psi_0\rangle$ is bounded. The invariance of   and the induction hypothesis imply that

$$\left| \langle\ ,m(\psi_1 \circ f^{n_1})\ldots(\psi_r \circ f^{n_r})\rangle - \prod_{i=0}^{r} \langle\ ,\psi_i\rangle \right|$$

$$= \left| \langle\ ,m\psi_1(\psi_2 \circ f^{n_2-n_1})\ldots(\psi_r \circ f^{n_r-n_1})\rangle - m\prod_{i=1}^{r} \langle\ ,\psi_i\rangle \right| \leq cd^{-n}$$

for some constant $c > 0$. In order to get the desired estimate, it is enough to show that

$$\left| \langle\ ,(\psi_0 - m)(\psi_1 \circ f^{n_1})\ldots(\psi_r \circ f^{n_r})\rangle \right| \leq cd^{-n}.$$

Observe that the operator $(f^n)^*$ acts on $L^p(\ )$ for $p \geq 1$ and its norm is bounded by 1. Using the invariance of   and Hölder's inequality, we get for $p := r+1$

$$\left| \langle\ ,(\psi_0 - m)(\psi_1 \circ f^{n_1})\ldots(\psi_r \circ f^{n_r})\rangle \right|$$

$$= \left| \langle\ ,\Lambda^{n_1}(\psi_0 - m)\psi_1\ldots(\psi_r \circ f^{n_r-n_1})\rangle \right|$$

$$\leq \|\Lambda^{n_1}(\psi_0 - m)\|_{L^p(\ )}\|\psi_1\|_{L^p(\ )}\ldots\|\psi_r \circ f^{n_r-n_1}\|_{L^p(\ )}$$

$$\leq cd^{-n_1}\|\psi_1\|_{L^p(\ )}\ldots\|\psi_r\|_{L^p(\ )},$$

for some constant $c > 0$. Since $\|\psi_i\|_{L^p(\ )} \lesssim \|\psi_i\|_{\mathrm{DSH}}$, the previous estimates imply the result. Note that as in Theorem 1.83, it is enough to assume that ψ_i is d.s.h. for $i \leq r-1$ and ψ_r is in $L^p(\ )$ for some $p > 1$. $\square$

The mixing of   implies that for any measurable observable φ, the times series $\varphi \circ f^n$, behaves like independent random variables with the same distribution. For example, the dependence of $\varphi \circ f^n$ and φ is weak when n is large: if a,b are real numbers, then the measure of $\{\varphi \circ f^n \leq a$ and $\varphi \leq b\}$ is almost equal to $\{\varphi \circ f^n \leq a\}$ $\{\varphi \leq b\}$. Indeed, it is equal to

$$\langle\ ,(\mathbf{1}_{]-\infty,a]} \circ \varphi \circ f^n)(\mathbf{1}_{]-\infty,b]} \circ \varphi)\rangle,$$

and when n is large, mixing implies that the last integral is approximatively equal to

$$\langle\ ,\mathbf{1}_{]-\infty,a]} \circ \varphi\rangle\langle\ ,\mathbf{1}_{]-\infty,b]} \circ \varphi\rangle =\ \{\varphi \leq a\}\ \{\varphi \leq b\} =\ \{\varphi \circ f^n \leq a\}\ \{\varphi \leq b\}.$$

The estimates on the decay of correlations obtained in the above results, give at which speed the observables become "almost independent". We are going to show that under weak assumptions on the regularity of observables φ, the times series $\varphi \circ f^n$, satisfies the Central Limit Theorem (CLT for short). We recall the classical CLT for independent random variables. In what follows, $\mathrm{E}(\cdot)$ denotes expectation, i.e. the mean, of a random variable.

Theorem 1.85. *Let $(X,\mathscr{F},\nu)$ be a probability space. Let $Z_1,Z_2,\ldots$ be independent identically distributed (i.i.d. for short) random variables with values in $\mathbb{R}$, and of*

mean zero, i.e. $E(Z_n) = 0$. *Assume also that* $0 < E(Z_n^2) < \infty$. *Then for any open interval* $I \subset \mathbb{R}$ *and for* $\sigma := E(Z_n^2)^{1/2}$, *we have*

$$\lim_{N \to \infty} v \left\{ \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} Z_n \in I \right\} = \frac{1}{\sqrt{2\pi}\sigma} \int_I e^{-\frac{t^2}{2\sigma^2}} \, dt.$$

The important hypothesis here is that the variables have the same distribution (this means that for every interval $I \subset \mathbb{R}$, $\langle v, \mathbf{1}_I \circ Z_n \rangle$ is independent of n, where $\mathbf{1}_I$ is the characteristic function of I) and that they are independent. The result can be phrased as follows. If we define the random variables $\widehat{Z}_N$ by

$$\widehat{Z}_N := \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} Z_n,$$

then the sequence of probability measures $(\widehat{Z}_N)_*(v)$ on $\mathbb{R}$ converges to the probability measure of density $\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}}$. This is also called *the convergence in law*.

We want to replace the random variables Z_n with the functions $\varphi \circ f^n$ on the probability space $(\mathbb{P}^k, \mathscr{B}, \ )$ where $\mathscr{B}$ is the Borel σ-algebra. The fact that is invariant means exactly that $\varphi \circ f^n$ are identically distributed. We state first a central limit theorem due to Gordin [G], see also [V]. For simplicity, we consider a measurable map $g : (X, \mathscr{F}) \to (X, \mathscr{F})$ as above. Define $\mathscr{F}_n := g^{-n}(\mathscr{F})$, $n \geq 0$, the σ-algebra of $g^{-n}(A)$, with $A \in \mathscr{F}$. This sequence is non-increasing. Denote by $E(\varphi | \mathscr{F}')$ the conditional expectation of φ with respect to a σ-algebra $\mathscr{F}' \subset \mathscr{F}$. We say that φ is *a coboundary* if $\varphi = \psi \circ g - \psi$ for some function $\psi \in L^2(v)$.

Theorem 1.86 (Gordin). *Let v be an ergodic invariant probability measure on X. Let φ be an observable in $L^1(v)$ such that $\langle v, \varphi \rangle = 0$. Suppose*

$$\sum_{n \geq 0} \|E(\varphi | \mathscr{F}_n)\|^2_{L^2(v)} < \infty.$$

Then $\langle v, \varphi^2 \rangle + 2 \sum_{n \geq 1} \langle v, \varphi(\varphi \circ g^n) \rangle$ is a finite positive number which vanishes if and only if φ is a coboundary. Moreover, if

$$\sigma := \left[\langle v, \varphi^2 \rangle + 2 \sum_{n \geq 1} \langle v, \varphi(\varphi \circ g^n) \rangle \right]^{1/2}$$

is strictly positive, then φ satisfies the central limit theorem with variance σ: for any interval $I \subset \mathbb{R}$

$$\lim_{N \to \infty} v \left\{ \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \varphi \circ g^n \in I \right\} = \frac{1}{\sqrt{2\pi}\sigma} \int_I e^{-\frac{t^2}{2\sigma^2}} \, dt.$$

It is not difficult to see that a function u is $\mathscr{F}_n$-measurable if and only if $u = u' \circ g^n$ with u' $\mathscr{F}$-measurable. Let $L^2(v, \mathscr{F}_n)$ denote the space of $\mathscr{F}_n$-measurable functions which are in $L^2(v)$. Then, $E(\varphi | \mathscr{F}_n)$ is the orthogonal projection of $\varphi \in L^2(v)$ into $L^2(v, \mathscr{F}_n)$.

A straightforward computation using the invariance of ν gives that the variance σ in the above theorem is equal to

$$\sigma = \lim_{n \to \infty} n^{-1/2} \|\varphi + \cdots + \varphi \circ g^{n-1}\|_{L^2(\nu)}.$$

When φ is orthogonal to all $\varphi \circ g^n$, we find that $\sigma = \|\varphi\|_{L^2(\ )}$. So, Gordin's theorem assumes a weak dependence and concludes that the observables satisfy the central limit theorem.

Consider now the dynamical system associated to an endomorphism f of $\mathbb{P}^k$ as above. Let $\mathscr{B}$ denote the Borel σ-algebra on $\mathbb{P}^k$ and define $\mathscr{B}_n := f^{-n}(\mathscr{B})$. Since the measure satisfies $f^*(\ ) = d^k$, the norms $\|\mathrm{E}(\cdot|\mathscr{B}_n)\|_{L^2(\ )}$ can be expressed in terms of the operator Λ. We have the following lemma.

Lemma 1.87. *Let φ be an observable in $L^2(\ )$. Then*

$$\mathrm{E}(\varphi|\mathscr{F}_n) = \Lambda^n(\varphi) \circ f^n \quad and \quad \|\mathrm{E}(\varphi|\mathscr{B}_n)\|_{L^p(\ )} = \|\Lambda^n(\varphi)\|_{L^p(\ )},$$

for $1 \le p \le 2$.

Proof. We have

$$\langle\ , \varphi(\psi \circ f^n)\rangle = \langle d^{-kn}(f^n)^*(\ ), \varphi(\psi \circ f^n)\rangle = \langle\ , d^{-kn}(f^n)_*[\varphi(\psi \circ f^n)]\rangle$$
$$= \langle\ , \Lambda^n(\varphi)\psi\rangle = \langle\ , [\Lambda^n(\varphi) \circ f^n][\psi \circ f^n]\rangle.$$

This proves the first assertion. The invariance of $\qquad$ implies that $\|\psi \circ f^n\|_{L^p(\ )} = \|\psi\|_{L^p(\ )}$. Therefore, the second assertion is a consequence of the first one. $\qquad\square$

Gordin's Theorem 1.86, Corollaries 1.41 and 1.42, applied to $q = 2$, give the following result.

Corollary 1.88. *Let f be an endomorphism of algebraic degree $d \ge 2$ of $\mathbb{P}^k$ and its equilibrium measure. Let φ be a d.s.h. function or a Hölder continuous function on $\mathbb{P}^k$, such that $\langle\ , \varphi\rangle = 0$. Assume that φ is not a coboundary. Then φ satisfies the central limit theorem with the variance $\sigma > 0$ given by*

$$\sigma^2 := \langle\ , \varphi^2\rangle + 2 \sum_{n \ge 1} \langle\ , \varphi(\varphi \circ f^n)\rangle.$$

We give an interesting decomposition of the space $L_0^2(\ )$ which shows that Λ, acts like a "generalized shift". Recall that $L_0^2(\ )$ is the space of functions $\psi \in L^2(\ )$ such that $\langle\ , \psi\rangle = 0$. Corollary 1.88 can also be deduced from the following result.

Proposition 1.89. *Let f be an endomorphism of algebraic degree $d \ge 2$ of $\mathbb{P}^k$ and the corresponding equilibrium measure. Define*

$$V_n := \{\psi \in L_0^2(\ ),\ \Lambda^n(\psi) = 0\}.$$

Then, we have $V_{n+1} = V_n \oplus V_1 \circ f^n$ as an orthogonal sum and $L_0^2(\ ) = \oplus_{n=0}^{\infty} V_1 \circ f^n$ as a Hilbert sum. Let $\psi = \sum \psi_n \circ f^n$, with $\psi_n \in V_1$, be a function in $L_0^2(\ )$.

Then, ψ satisfies the Gordin's condition, see Theorem 1.86, if and only if the sum $\sum_{n\geq1} n\|\psi_n\|^2_{L^2(\ )}$ is finite. Moreover, if ψ is d.s.h. (resp. of class $\mathscr{C}^\nu$, $0 < \nu \leq 2$) with $\langle\ ,\psi\rangle = 0$, then $\|\psi_n\|_{L^2(\ )} \lesssim d^{-n}$ (resp. $\|\psi_n\|_{L^2(\ )} \lesssim d^{-n\nu/2}$).

Proof. It is easy to check that $V_n^\perp = \{\theta \circ f^n,\ \theta \in L^2(\ )\}$. Let W_{n+1} denote the orthogonal complement of V_n in V_{n+1}. Suppose $\theta \circ f^n$ is in W_{n+1}. Then, $\Lambda(\theta) = 0$. This gives the first decomposition in the proposition.

For the second decomposition, observe that $\oplus_{n=0}^\infty V_1 \circ f^n$ is a direct orthogonal sum. We only have to show that $\cup V_n$ is dense in $L^2_0(\ )$. Let θ be an element in $\cap V_n^\perp$. We have to show that $\theta = 0$. For every n, $\theta = \theta_n \circ f^n$ for appropriate θ_n. Hence, θ is measurable with respect to the σ-algebra $\mathscr{B}_\infty := \cap_{n\geq0}\mathscr{B}_n$. We show that $\mathscr{B}_\infty$ is the trivial algebra. Let A be an element of $\mathscr{B}_\infty$. Define $A_n = f^n(A)$. Since A is in $\mathscr{B}_\infty$, $\mathbf{1}_A = \mathbf{1}_{A_n} \circ f^n$ and $\Lambda^n(\mathbf{1}_A) = \mathbf{1}_{A_n}$. K-mixing implies that $\Lambda^n(\mathbf{1}_A)$ converges in $L^2(\ )$ to a constant, see Theorem 1.82. So, $\mathbf{1}_{A_n}$ converges to a constant which is necessarily 0 or 1. We deduce that $\ (A_n)$ converges to 0 or 1. On the other hand, we have

$$(A_n) = \langle\ ,\mathbf{1}_{A_n}\rangle = \langle\ ,\mathbf{1}_{A_n}\circ f^n\rangle = \langle\ ,\mathbf{1}_A\rangle = \ (A).$$

Therefore, A is of measure 0 or 1. This implies the decomposition of $L^2_0(\ )$.

Suppose now that $\psi := \sum \psi_n \circ f^n$ with $\Lambda(\psi_n) = 0$, is an element of $L^2_0(\ )$. We have $\mathrm{E}(\psi|\mathscr{B}_n) = \sum_{i\geq n}\psi_i \circ f^i$. So,

$$\sum_{n\geq0}\|\mathrm{E}(\psi|\mathscr{B}_n)\|^2_{L^2(\ )} = \sum_{n\geq0}(n+1)\|\psi_n\|^2_{L^2(\ )},$$

and ψ satisfies Gordin's condition if and only if the last sum is finite.

Let ψ be a d.s.h. function with $\langle\ ,\psi\rangle = 0$. It follows from Theorem 1.83 that

$$\|\mathrm{E}(\psi|\mathscr{B}_n)\|_{L^2(\ )} = \sup_{\|\varphi\|_{L^2(\ )}\leq1}|\langle\ ,(\varphi\circ f^n)\psi\rangle| \lesssim d^{-n}.$$

Since $\psi_n \circ f^n = \mathrm{E}(\psi|\mathscr{B}_n) - \mathrm{E}(\psi|\mathscr{B}_{n+1})$, the above estimate implies that

$$\|\psi_n\|_{L^2(\ )} = \|\psi_n\circ f^n\|_{L^2(\ )} \lesssim d^{-n}.$$

The case of $\mathscr{C}^\nu$ observables is proved in the same way.

Observe that if $(\psi_n \circ f^n)_{n\geq0}$ is the sequence of projections of ψ on the factors of the direct sum $\oplus_{n=0}^\infty V_1 \circ f^n$, then the coordinates of $\Lambda(\psi)$ are $(\psi_n \circ f^{n-1})_{n\geq1}$. $\square$

We continue the study with other types of convergence. Let us recall the almost sure version of the central limit theorem in probability theory. Let Z_n be random variables, identically distributed in $L^2(X,\mathscr{F},\nu)$, such that $\mathrm{E}(Z_n) = 0$ and $\mathrm{E}(Z_n^2) = \sigma^2$, $\sigma > 0$. We say that *the almost sure central limit theorem* holds if at ν-almost every point in X, the sequence of measures

$$\frac{1}{\log N}\sum_{n=1}^N \frac{1}{n}\delta_{n^{-1/2}\sum_{i=0}^{n-1}Z_i}$$

converges in law to the normal distribution of mean 0 and variance σ. In particular, v-almost surely

$$\frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \mathbf{1}_{\left\{n^{-1/2} \sum_{i=0}^{n-1} Z_i \leq t_0\right\}} \rightarrow \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{t_0} e^{-\frac{t^2}{2\sigma^2}} dt,$$

for any $t_0 \in \mathbb{R}$. In the central limit theorem, we only get the v-measure of the set $\left\{N^{-1/2} \sum_{n=0}^{N-1} Z_n < t_0\right\}$ when N goes to infinity. Here, we get an information at v-almost every point for the logarithmic averages.

The almost sure central limit theorem can be deduced from the so-called almost sure invariance principle (ASIP for short). In the case of i.i.d. random variables as above, this principle compares the variables $\widehat{Z}_N$ with Brownian motions and gives some information about the fluctuations of $\widehat{Z}_N$ around 0.

Theorem 1.90. *Let $(X, \mathscr{F}, v)$ be a probability space. Let (Z_n) be a sequence of i.i.d. random variables with mean 0 and variance $\sigma > 0$. Assume that there is an $\alpha > 0$ such that Z_n is in $L^{2+\alpha}(v)$. Then, there is another probability space $(X', \mathscr{F}', v')$ with a sequence of random variables S'_N on X' which has the same joint distribution as $S_N := \sum_{n=0}^{N-1} Z_n$, and a Brownian motion B of variance σ on X' such that*

$$|S'_N - \mathsf{B}(N)| \leq cN^{1/2-\delta},$$

for some positive constants c, δ. It follows that

$$|N^{-1/2}S'_N - \mathsf{B}(1)| \leq cN^{-\delta}.$$

For weakly dependent variables, this type of result is a consequence of a theorem due to Philipp-Stout [PS]. It gives conditions which imply that the ASIP holds. Lacey-Philipp proved in [LP] that the ASIP implies the almost sure central limit theorem. For holomorphic endomorphisms of $\mathbb{P}^k$, we have the following result due to Dupont which holds in particular for Hölder continuous observables [DP2].

Theorem 1.91. *Let f be an endomorphism of algebraic degree $d \geq 2$ as above and its equilibrium measure. Let φ be an observable with values in $\mathbb{R} \cup \{-\infty\}$ such that e^φ is Hölder continuous, $H := \{\varphi = -\infty\}$ is an analytic set and $|\varphi| \lesssim |\log \operatorname{dist}(\cdot, H)|^\rho$ near H for some $\rho > 0$. If $\langle \ , \varphi \rangle = 0$ and φ is not a coboundary, then the almost sure invariance principle holds for φ. In particular, the almost sure central limit theorem holds for such observables.*

The ASIP in the above setting says that there is a probability space $(X', \mathscr{F}', v')$ with a sequence of random variables S'_N on X which has the same joint distribution as $S_N := \sum_{n=0}^{N-1} \varphi \circ f^n$, and a Brownian motion B of variance σ on X' such that

$$|S'_N - \mathsf{B}(N)| \leq cN^{1/2-\delta},$$

for some positive constants c, δ.

The ASIP implies other stochastic results, see [PS], in particular, *the law of the iterated logarithm*. With our notations, it implies that for φ as above

$$\limsup_{N \to \infty} \frac{S_N(\varphi)}{\sigma \sqrt{N \log \log(N \sigma^2)}} = 1 \qquad \text{-almost everywhere.}$$

Dupont's approach is based on the Philipp-Stout's result applied to a Bernoulli system and a quantitative Bernoulli property of the equilibrium measure of f, i.e. a construction of a coding tree. We refer to Dupont and Przytycki-Urbanski-Zdunik [PU] for these results.

Recall the Bernoulli property of   which was proved by Briend in [BJ1]. The dimension one case is due to Mañé [MA] and Heicklen-Hoffman [HH]. Denote by (Σ, ν, σ) the *one-sided d^k-shift*, where $\Sigma := \{1, \ldots, d^k\}^{\mathbb{N}}$, ν is the probability measure on Σ induced by the equilibrium probability measure on $\{1, \ldots, d^k\}$ and $\sigma : \Sigma \to \Sigma$ is the d^k to 1 map defined by $\sigma(\alpha_0, \alpha_1, \ldots) = (\alpha_1, \alpha_2, \ldots)$.

Theorem 1.92. *Let f and   be as above. Then $(\mathbb{P}^k, , f)$ is measurably conjugated to (Σ, ν, σ). More precisely, there is a measurable map $\pi : \Sigma \to \mathbb{P}^k$, defined out of a set of zero ν-measure, which is invertible out of a set of zero -measure, such that $\pi_*(\nu) = $ and $f = \pi \circ \sigma \circ \pi^{-1}$ -almost everywhere.*

The proof uses a criterion, the so called *tree very weak Bernoulli* property (tree-vwB for short) due to Hoffman-Rudolph [HR]. One can use Proposition 1.51 in order to check this criterion.

The last stochastic property we consider here is the large deviations theorem. As above, we first recall the classical result in probability theory.

Theorem 1.93. *Let $Z_1, Z_2, \ldots$ be independent random variables on $(X, \mathscr{F}, \nu)$, identically distributed with values in $\mathbb{R}$, and of mean zero, i.e. $E(Z_1) = 0$. Assume also that for $t \in \mathbb{R}$, $\exp(tZ_n)$ is integrable. Then, the limit*

$$I(\varepsilon) := - \lim_{N \to \infty} \log \nu \left\{ \left| \frac{Z_1 + \cdots + Z_N}{N} \right| > \varepsilon \right\}.$$

exists and $I(\varepsilon) > 0$ for $\varepsilon > 0$.

The theorem estimates the size of the set where the average is away from zero, the expected value. We have

$$\nu \left\{ \left| \frac{Z_1 + \cdots + Z_N}{N} \right| > \varepsilon \right\} \sim e^{-NI(\varepsilon)}.$$

Our goal is to give an analogue for the equilibrium measure of endomorphisms of $\mathbb{P}^k$. We first prove an abstract result corresponding to the above Gordin's result for the central limit theorem.

Consider a dynamical system $g : (X, \mathscr{F}, \nu) \to (X, \mathscr{F}, \nu)$ as above where ν is an invariant probability measure. So, g^* defines a linear operator of norm 1 from $L^2(\nu)$ into itself. We say that g has *bounded Jacobian* if there is a constant $\kappa > 0$ such that $\nu(g(A)) \leq \kappa \nu(A)$ for every $A \in \mathscr{F}$. The following result was obtained in [DNS].

Theorem 1.94. *Let $g : (X, \mathscr{F}, \nu) \to (X, \mathscr{F}, \nu)$ be a map with bounded Jacobian as above. Define $\mathscr{F}_n := g^{-n}(\mathscr{F})$. Let ψ be a bounded real-valued measurable function. Assume there are constants $\delta > 1$ and $c > 0$ such that*

$$\left\langle \nu, e^{\delta^n |\mathrm{E}(\psi|\mathscr{F}_n) - \langle \nu, \psi \rangle|} \right\rangle \leq c \quad \textit{for every} \quad n \geq 0.$$

Then ψ satisfies a weak large deviations theorem. More precisely, for every $\varepsilon > 0$, there exists a constant $h_\varepsilon > 0$ such that

$$\nu \left\{ x \in X : \left| \frac{1}{N} \sum_{n=0}^{N-1} \psi \circ g^n(x) - \langle \nu, \psi \rangle \right| > \varepsilon \right\} \leq e^{-N(\log N)^{-2} h_\varepsilon}$$

for all N large enough[6].

We first prove some preliminary lemmas. The following one is a version of the classical Bennett's inequality see [DZ, Lemma 2.4.1].

Lemma 1.95. *Let ψ be an observable such that $\|\psi\|_{L^\infty(\nu)} \leq b$ for some constant $b \geq 0$, and $\mathrm{E}(\psi) = 0$. Then*

$$\mathrm{E}(e^{\lambda \psi}) \leq \frac{e^{-\lambda b} + e^{\lambda b}}{2}$$

for every $\lambda \geq 0$.

Proof. We can assume $\lambda = 1$. Consider first the case where there is a measurable set A such that $\nu(A) = 1/2$. Let ψ_0 be the function which is equal to $-b$ on A and to b on $X \setminus A$. We have $\psi_0^2 = b^2 \geq \psi^2$. Since $\nu(A) = 1/2$, we have $\mathrm{E}(\psi_0) = 0$. Let $g(t) = a_0 t^2 + a_1 t + a_2$, be the unique quadratic function such that $h(t) := g(t) - e^t$ satisfies $h(b) = 0$ and $h(-b) = h'(-b) = 0$. We have $g(\psi_0) = e^{\psi_0}$.

Since $h''(t) = 2a_0 - e^t$ admits at most one zero, h' admits at most two zeros. The fact that $h(-b) = h(b) = 0$ implies that h' vanishes in $]-b, b[$. Hence h' admits exactly one zero at $-b$ and another one in $]-b, b[$. We deduce that h'' admits a zero. This implies that $a_0 > 0$. Moreover, h vanishes only at $-b$, b and $h'(b) \neq 0$. It follows that $h(t) \geq 0$ on $[-b, b]$ because h is negative near $+\infty$. Thus, $e^t \leq g(t)$ on $[-b, b]$ and then $e^\psi \leq g(\psi)$.

Since $a_0 > 0$, if an observable ϕ satisfies $\mathrm{E}(\phi) = 0$, then $\mathrm{E}(g(\phi))$ is an increasing function of $\mathrm{E}(\phi^2)$. Now, using the properties of ψ and ψ_0, we obtain

$$\mathrm{E}(e^\psi) \leq \mathrm{E}(g(\psi)) \leq \mathrm{E}(g(\psi_0)) = \mathrm{E}(e^{\psi_0}) = \frac{e^{-b} + e^b}{2}.$$

This completes the proof under the assumption that $\nu(A) = 1/2$ for some measurable set A.

The general case is deduced from the previous particular case. Indeed, it is enough to apply the first case to the disjoint union of $(X, \mathscr{F}, \nu)$ with a copy

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[6] In the LDT for independent random variables, there is no factor $(\log N)^{-2}$ in the estimate.

$(X',\mathscr{F}',v')$ of this space, i.e. to the space $(X\cup X',\mathscr{F}\cup\mathscr{F}',\frac{v}{2}+\frac{v'}{2})$, and to the function equal to ψ on X and on X'. $\qquad\square$

Lemma 1.96. *Let* ψ *be an observable such that* $\|\psi\|_{L^{\infty}(v)}\le b$ *for some constant* $b\ge 0$, *and* $E(\psi|\mathscr{F}_1)=0$. *Then*

$$E(e^{\lambda\psi}|\mathscr{F}_1)\le\frac{e^{-\lambda b}+e^{\lambda b}}{2}$$

for every $\lambda\ge 0$.

Proof. We consider the desintegration of v with respect to g. For v-almost every $x\in X$, there is a positive measure v_x on $g^{-1}(x)$ such that if φ is a function in $L^1(v)$ then

$$\langle v,\varphi\rangle=\int_X\langle v_x,\varphi\rangle dv(x).$$

Since v is g-invariant, we have

$$\langle v,\varphi\rangle=\langle v,\varphi\circ g\rangle=\int_X\langle v_x,\varphi\circ g\rangle dv(x)=\int_X\|v_x\|\varphi(x)dv(x).$$

Therefore, v_x is a probability measure for v-almost every x. Using also the invariance of v, we obtain for φ and ϕ in $L^2(v)$ that

$$\langle v,\varphi(\phi\circ g)\rangle=\int_X\langle v_x,\varphi(\phi\circ g)\rangle dv(x)=\int_X\langle v_x,\varphi\rangle\phi(x)dv(x)$$
$$=\int_X\langle v_{g(x)},\varphi\rangle\phi(g(x))dv(x).$$

We deduce that
$$E(\varphi|\mathscr{F}_1)(x)=\langle v_{g(x)},\varphi\rangle.$$

So, the hypothesis in the lemma is that $\langle v_x,\psi\rangle=0$ for v-almost every x. It suffices to check that
$$\langle v_x,e^{\lambda\psi}\rangle\le\frac{e^{-\lambda b}+e^{\lambda b}}{2}.$$

But this is a consequence of Lemma 1.95 applied to v_x instead of v. $\qquad\square$

We continue the proof of Theorem 1.94. Without loss of generality we can assume that $\langle v,\psi\rangle=0$ and $|\psi|\le 1$. The general idea is to write $\psi=\psi'+(\psi''-\psi''\circ g)$ for functions ψ' and ψ'' in $L^2(v)$ such that

$$E(\psi'\circ g^n|\mathscr{F}_{n+1})=0,\qquad n\ge 0.$$

In the language of probability theory, these identities mean that $(\psi'\circ g^n)_{n\ge 0}$ is a *reversed martingale difference* as in Gordin's approach, see also [V]. The strategy is to prove the weak LDT for ψ' and for the coboundary $\psi''-\psi''\circ g$. Theorem 1.94 is then a consequence of Lemmas 1.99 and 1.101 below.

Let Λ_g denote the adjoint of the operator $\varphi \mapsto \varphi \circ g$ on $L^2(\nu)$. These operators are of norm 1. The computation in Lemma 1.96 shows that $E(\varphi|\mathscr{F}_1) = \Lambda_g(\varphi) \circ g$. We obtain in the same way that $E(\varphi|\mathscr{F}_n) = \Lambda_g^n(\varphi) \circ g^n$. Define

$$\psi'' := -\sum_{n=1}^{\infty} \Lambda_g^n(\psi), \qquad \psi' := \psi - (\psi'' - \psi'' \circ g).$$

Using the hypotheses in Theorem 1.94, we see that ψ' and ψ'' are in $L^2(\nu)$ with norms bounded by some constant. However, we loose the uniform boundedness: these functions are not necessarily in $L^{\infty}(\nu)$.

Lemma 1.97. *We have $\Lambda_g^n(\psi') = 0$ for $n \geq 1$ and $E(\psi' \circ g^n|\mathscr{F}_m) = 0$ for $m > n \geq 0$.*

Proof. Clearly $\Lambda_g(\psi'' \circ g) = \psi''$. We deduce from the definition of ψ'' that

$$\Lambda_g(\psi') = \Lambda_g(\psi) - \Lambda_g(\psi'') + \Lambda_g(\psi'' \circ g) = \Lambda_g(\psi) - \Lambda_g(\psi'') + \psi'' = 0.$$

Hence, $\Lambda_g^n(\psi') = 0$ for $n \geq 1$. For every function ϕ in $L^2(\nu)$, since ν is invariant, we have for $m > n$

$$\langle \nu, (\psi' \circ g^n)(\phi \circ g^m) \rangle = \langle \nu, \psi'(\phi \circ g^{m-n}) \rangle = \langle \nu, \Lambda_g^{m-n}(\psi')\phi \rangle = 0.$$

It follows that $E(\psi' \circ g^n|\mathscr{F}_m) = 0$. $\qquad\square$

Lemma 1.98. *There are constants $\delta_0 > 1$ and $c > 0$ such that*

$$\nu\{|\psi'| > b\} \leq ce^{-\delta_0^b} \quad and \quad \nu\{|\psi''| > b\} \leq ce^{-\delta_0^b}$$

for any $b \geq 0$. In particular, $t\psi'$ and $t\psi''$ are ν-integrable for every $t \geq 0$.

Proof. Since $\psi' := \psi - (\psi'' - \psi'' \circ g)$ and ψ is bounded, it is enough to prove the estimate on ψ''. Indeed, the invariance of ν implies that $\psi'' \circ g$ satisfies a similar inequality.

Fix a positive constant δ_1 such that $1 < \delta_1^2 < \delta$, where δ is the constant in Theorem 1.94. Define $\varphi := \sum_{n \geq 1} \delta_1^{2n} |\Lambda_g^n(\psi)|$. We first show that there is a constant $\alpha > 0$ such that $\nu\{\varphi \geq b\} \lesssim e^{-\alpha b}$ for every $b \geq 0$. Recall that $E(\psi|\mathscr{F}_n) = \Lambda_g^n(\psi) \circ g^n$. Using the hypothesis of Theorem 1.94, the inequality $\sum \frac{1}{2n^2} \leq 1$ and the invariance of ν, we obtain for $b \geq 0$

$$\nu\{\varphi \geq b\} \leq \sum_{n \geq 1} \nu\left\{|\Lambda_g^n(\psi)| \geq \frac{\delta_1^{-2n}b}{2n^2}\right\} \leq \sum_{n \geq 1} \nu\left\{|E(\psi|\mathscr{F}_n)| \geq \frac{\delta_1^{-2n}b}{2n^2}\right\}$$

$$= \sum_{n \geq 1} \nu\left\{\delta^n|E(\psi|\mathscr{F}_n)| \geq \frac{\delta^n\delta_1^{-2n}b}{2n^2}\right\} \lesssim \sum_{n \geq 1} \exp\left(\frac{-\delta^n\delta_1^{-2n}b}{2n^2}\right).$$

It follows that $\nu\{\varphi \geq b\} \lesssim e^{-\alpha b}$ for some constant $\alpha > 0$.

We prove now the estimate $v\{|\psi''| > b\} \le ce^{-\delta_0^b}$. It is enough to consider the case where $b = 2l$ for some positive integer l. Recall that for simplicity we assumed $|\psi| \le 1$. It follows that $|E(\psi|\mathscr{F}_n)| \le 1$ and hence $|\Lambda_g^n(\psi)| \le 1$. We have

$$|\psi''| \le \sum_{n\ge 1}|\Lambda_g^n(\psi)| \le \delta_1^{-2l}\sum_{n\ge 1}\delta_1^{2n}|\Lambda_g^n(\psi)| + \sum_{1\le n\le l}|\Lambda_g^n(\psi)| \le \delta_1^{-2l}\varphi + l.$$

Consequently,

$$v\{|\psi''| > 2l\} \le v\{\varphi > \delta_1^{2l}\} \lesssim e^{-\alpha\delta_1^{2l}}.$$

It is enough to choose $\delta_0 < \delta_1$ and c large enough. $\qquad\square$

Lemma 1.99. *The coboundary $\psi'' - \psi'' \circ g$ satisfies the LDT.*

Proof. Given a function $\phi \in L^1(\ )$, recall that Birkhoff's sum $S_N(\phi)$ is defined by

$$S_0(\phi) := 0 \qquad \text{and} \qquad S_N(\phi) := \sum_{n=0}^{N-1}\phi \circ g^n \quad \text{for } N \ge 1.$$

Observe that $S_N(\psi'' - \psi'' \circ g) = \psi'' - \psi'' \circ g^N$. Consequently, for a given $\varepsilon > 0$, using the invariance of v, we have

$$v\{|S_N(\psi'' - \psi'' \circ g)| > N\varepsilon\} \le v\left\{|\psi'' \circ g^N| > \frac{N\varepsilon}{2}\right\} + \left\{|\psi''| > \frac{N\varepsilon}{2}\right\}$$

$$= 2v\left\{|\psi''| > \frac{N\varepsilon}{2}\right\}.$$

Lemma 1.98 implies that the last expression is smaller than e^{-Nh_ε} for some $h_\varepsilon > 0$ and for N large enough. This completes the proof. $\qquad\square$

It remains to show that ψ' satisfies the weak LDT. We use the following lemma.

Lemma 1.100. *For every $b \ge 1$, there are Borel sets W_N such that $v(W_N) \le cNe^{-\delta_0^b}$ and*

$$\int_{X\setminus W_N}e^{\lambda S_N(\psi')}dv \le 2\left[\frac{e^{-\lambda b} + e^{\lambda b}}{2}\right]^N,$$

where $c > 0$ is a constant independent of b.

Proof. For $N = 1$, define $W := \{|\psi'| > b\}$, $W' := g(W)$ and $W_1 := g^{-1}(W')$. Recall that the Jacobian of v is bounded by some constant κ. This and Lemma 1.98 imply that

$$v(W_1) = v(W') = v(g(W)) \le \kappa v(W) \le ce^{-\delta_0^b}$$

for some constant $c > 0$. We also have

$$\int_{X\setminus W_1}e^{\lambda S_1(\psi')}dv = \int_{X\setminus W_1}e^{\lambda \psi'}dv \le e^{\lambda b} \le 2\left[\frac{e^{-\lambda b} + e^{\lambda b}}{2}\right].$$

So, the lemma holds for $N = 1$.

Suppose the lemma for $N \geq 1$, we prove it for $N+1$. Define

$$W_{N+1} := g^{-1}(W_N) \cup W_1 = g^{-1}(W_N \cup W').$$

We have

$$v(W_{N+1}) \leq v(g^{-1}(W_N)) + v(W_1) = v(W_N) + v(W_1) \leq c(N+1)e^{-\delta_0^b}.$$

We will apply Lemma 1.96 to the function ψ^* such that $\psi^* = \psi'$ on $X \setminus W_1$ and $\psi^* = 0$ on W_1. By Lemma 1.97, we have $E(\psi^*|\mathscr{F}_1) = 0$ since W_1 is an element of $\mathscr{F}_1$. The choice of W_1 gives that $|\psi^*| \leq b$. By Lemma 1.96, we have

$$E(e^{\lambda \psi^*}|\mathscr{F}_1) \leq \frac{e^{-\lambda b} + e^{\lambda b}}{2} \quad \text{on} \quad X \quad \text{for} \quad \lambda \geq 0.$$

It follows that

$$E(e^{\lambda \psi'}|\mathscr{F}_1) \leq \frac{e^{-\lambda b} + e^{\lambda b}}{2} \quad \text{on} \quad X \setminus W_1 \quad \text{for} \quad \lambda \geq 0.$$

Now, using the fact that W_{N+1} and $e^{\lambda S_N(\psi' \circ g)}$ are $\mathscr{F}_1$-measurable, we can write

$$\int_{X \setminus W_{N+1}} e^{\lambda S_{N+1}(\psi')} dv = \int_{X \setminus W_{N+1}} e^{\lambda \psi'} e^{\lambda S_N(\psi' \circ g)} dv$$

$$= \int_{X \setminus W_{N+1}} E(e^{\lambda \psi'}|\mathscr{F}_1) e^{\lambda S_N(\psi' \circ g)} dv.$$

Since $W_{N+1} = g^{-1}(W_N) \cup W_1$, the last integral is bounded by

$$\sup_{X \setminus W_1} E(e^{\lambda \psi'}|\mathscr{F}_1) \int_{X \setminus g^{-1}(W_N)} e^{\lambda S_N(\psi' \circ g)} dv$$

$$\leq \left[\frac{e^{-\lambda b} + e^{\lambda b}}{2} \right] \int_{X \setminus W_N} e^{\lambda S_N(\psi')} dv$$

$$\leq 2 \left[\frac{e^{-\lambda b} + e^{\lambda b}}{2} \right]^{N+1},$$

where the last inequality follows from the induction hypothesis. So, the lemma holds for $N+1$. □

The following lemma, together with Lemma 1.99, implies Theorem 1.94.

Lemma 1.101. *The function ψ' satisfies the weak LDT.*

Proof. Fix an $\varepsilon > 0$. By Lemma 1.100, we have, for every $\lambda \geq 0$

$$v\{S_N(\psi') \geq N\varepsilon\} \leq v(W_N) + e^{-\lambda N\varepsilon} \int_{X \setminus W_N} e^{\lambda S_N(\psi')} dv$$

$$\leq cN e^{-\delta_0^b} + 2e^{-\lambda N\varepsilon} \left[\frac{e^{-\lambda b} + e^{\lambda b}}{2} \right]^N.$$

Let $b := \log N (\log \delta_0)^{-1}$. We have

$$cNe^{-\delta_0^b} = cNe^{-N} \le e^{-N/2}$$

for N large. We also have

$$\frac{e^{-\lambda b} + e^{\lambda b}}{2} = \sum_{n \ge 0} \frac{\lambda^{2n} b^{2n}}{(2n)!} \le e^{\lambda^2 b^2}.$$

Therefore, if $\lambda := u \varepsilon b^{-2}$ with a fixed $u > 0$ small enough

$$2e^{-\lambda N \varepsilon} \left[\frac{e^{-\lambda b} + e^{\lambda b}}{2} \right]^N \le 2e^{-\varepsilon^2 b^{-2}(1-u)Nu} = 2e^{-2N(\log N)^{-2} h_\varepsilon}$$

for some constant $h_\varepsilon > 0$. We deduce from the previous estimates that

$$\nu \left\{ \mathsf{S}_N(\psi') \ge N \varepsilon \right\} \le e^{-N(\log N)^{-2} h_\varepsilon}$$

for N large. A similar estimate holds for $-\psi'$. So, ψ' satisfies the weak LDT. $\qquad \square$

We deduce from Theorem 1.94, Corollaries 1.41 and 1.42 the following result [DNS].

Theorem 1.102. *Let f be a holomorphic endomorphism of $\mathbb{P}^k$ of algebraic degree $d \ge 2$. Then the equilibrium measure of f satisfies the weak large deviations theorem for bounded d.s.h. observables and also for Hölder continuous observables. More precisely, if a function ψ is bounded d.s.h. or Hölder continuous, then for every $\varepsilon > 0$ there is a constant $h_\varepsilon > 0$ such that*

$$\left\{ z \in \mathbb{P}^k : \left| \frac{1}{N} \sum_{n=0}^{N-1} \psi \circ f^n(z) - \langle \ , \psi \rangle \right| > \varepsilon \right\} \le e^{-N(\log N)^{-2} h_\varepsilon}$$

for all N large enough.

The exponential estimate on $\Lambda^n(\psi)$ is crucial in the proofs of the previous results. It is nearly an estimate in sup-norm. Note that if $\|\Lambda^n(\psi)\|_{L^\infty(\ )}$ converge exponentially fast to 0 then ψ satisfies the LDT. This is the case for Hölder continuous observables in dimension 1, following a result by Drasin-Okuyama [DO], and when f is a generic map in higher dimension, see Remark 1.71. The LDT was recently obtained in dimension 1 by Xia-Fu in [X] for Lipschitz observables.

Exercise 1.103. Let $g : X \to X$ be a continuous map on a compact metric space X. Deduce from Birkhoff's theorem that any ergodic invariant measure of g is a limit of

$$\frac{1}{N} \sum_{n=0}^{N-1} \delta_{g^n(x)}$$

for an appropriate x.

Exercise 1.104. Show that if a Borel set A satisfies $(A) > 0$, then $(f^n(A))$ converges to 1.

Exercise 1.105. Show that $\sup_\psi I_n(\varphi, \psi)$ with ψ smooth $\|\psi\|_\infty \leq 1$ is equal to $\|\varphi\|_{L^1(\ )}$. Deduce that there is no decay of correlations which is uniform on $\|\psi\|_\infty$.

Exercise 1.106. Let $V_1 := \{\psi \in L^2_0(\ ), \Lambda(\psi) = 0\}$. Show that V_1 is infinite dimensional and that bounded functions in V_1 are dense in V_1 with respect to the $L^2(\ )$-topology. Using Theorem 1.82, show that the only eigenvalues of Λ are 0 and 1.

Exercise 1.107. Let φ be a d.s.h. function as in Corollary 1.88. Show that

$$\|\varphi + \cdots + \varphi \circ f^{n-1}\|^2_{L^2(\ )} - n\sigma^2 + \gamma = O(d^{-n}),$$

where $\gamma := 2\sum_{n \geq 1} n\langle\ , \varphi(\varphi \circ f^n)\rangle$ is a finite constant. Prove an analogous property for φ Hölder continuous.

1.7 Entropy, Hyperbolicity and Dimension

There are various ways to describe the complexity of a dynamical system. A basic measurement is the entropy which is closely related to the volume growth of the images of subvarieties. We will compute the topological entropy and the metric entropy of holomorphic endomorphisms of $\mathbb{P}^k$. We will also estimate the Lyapounov exponents with respect to the measure of maximal entropy and the Hausdorff dimension of this measure.

We recall few notions. Let (X, dist) be a compact metric space where dist is a distance on X. Let $g : X \to X$ be a continuous map. We introduce *the Bowen metric* associated to g. For a positive integer n, define the distance dist_n on X by

$$\text{dist}_n(x, y) := \sup_{0 \leq i \leq n-1} \text{dist}(g^i(x), g^i(y)).$$

We have $\text{dist}_n(x, y) > \varepsilon$ if the orbits $x, g(x), g^2(x), \ldots$ of x and $y, g(y), g^2(y), \ldots$ of y are distant by more than ε at a time i less than n. In which case, we say that x, y are (n, ε)-*separated*.

The topological entropy measures the rate of growth in function of time n, of the number of orbits that can be distinguished at ε-resolution. In other words, it measures the divergence of the orbits. More precisely, for $K \subset X$, not necessarily invariant, let $N(K, n, \varepsilon)$ denote the maximal number of points in K which are pairwise (n, ε)-separated. This number increases as ε decreases. *The topological entropy* of g on K is

$$h_t(g, K) := \sup_{\varepsilon > 0} \limsup_{n \to \infty} \frac{1}{n} \log N(K, n, \varepsilon).$$

The topological entropy of g is the entropy on X and is denoted by $h_t(g)$. The reader can check that if g is an isometry, then $h_t(g) = 0$. In complex dynamics, we often have that for ε small enough, $\frac{1}{n} \log N(X, n, \varepsilon)$ converge to $h_t(g)$.

Let f be an endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$ as above. As we have seen, the iterate f^n of f has algebraic degree d^n. If Z is an algebraic set in $\mathbb{P}^k$ of codimension p then the degree of $f^{-n}(Z)$, counted with multiplicity, is equal to $d^{pn} \deg(Z)$ and the degree of $f^n(Z)$, counting with multiplicity, is equal to $d^{(k-p)n} \deg(Z)$. This is a consequence of Bézout's theorem. Recall that the degree of an algebraic set of codimension p in $\mathbb{P}^k$ is the number of points of intersection with a generic projective subspace of dimension p.

The pull-back by f induces a linear map $f^* : H^{p,p}(\mathbb{P}^k, \mathbb{C}) \to H^{p,p}(\mathbb{P}^k, \mathbb{C})$ which is just the multiplication by d^p. The constant d^p is *the dynamical degree of order p* of f. Dynamical degrees were considered by Gromov in [GR] where he introduced a method to bound the topological entropy from above. We will see that they measure the volume growth of the graphs. The degree of maximal order d^k is also called *the topological degree*. It is equal to the number of points in a fiber counting with multiplicity. The push-forward by f^n induces a linear map $f_* : H^{p,p}(\mathbb{P}^k, \mathbb{C}) \to H^{p,p}(\mathbb{P}^k, \mathbb{C})$ which is the multiplication by d^{k-p}. These operations act continuously on positive closed currents and hence, the actions are compatible with cohomology, see Appendix A.1.

We have the following result due to Gromov [GR] for the upper bound and to Misiurewicz-Przytycky [MP] for the lower bound of the entropy.

Theorem 1.108. *Let f be a holomorphic endomorphism of algebraic degree d on $\mathbb{P}^k$. Then the topological entropy $h_t(f)$ of f is equal to $k \log d$, i.e. to the logarithm of the maximal dynamical degree.*

The inequality $h_t(f) \geq k \log d$ is a consequence of the following result which is valid for arbitrary $\mathscr{C}^1$ maps [MP].

Theorem 1.109 (Misuriewicz-Przytycki). *Let X be a compact smooth orientable manifold and $g : X \to X$ a $\mathscr{C}^1$ map. Then*

$$h_t(g) \geq \log |\deg(g)|.$$

Recall that *the degree* of g is defined as follows. Let Ω be a continuous form of maximal degree on X such that $\int_X \Omega \neq 0$. Then

$$\deg(g) := \frac{\int_X g^*(\Omega)}{\int_X \Omega}.$$

The number is independent of the choice of Ω. When X is a complex manifold, it is necessarily orientable and $\deg(g)$ is just the generic number of preimages of a point, i.e. the topological degree of g. In our case, the topological degree of f is equal to d^k. So, $h_t(f) \geq k \log d$.

Instead of using Misuriewicz-Przytycki theorem, it is also possible to apply the following important result due to Yomdin [YO].

Theorem 1.110 (Yomdin). *Let X be a compact smooth manifold and $g : X \to X$ a smooth map. Let Y be a manifold in X smooth up to the boundary, then*

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathrm{volume}(g^n(Y)) \leq h_t(g),$$

where the volume of $g^n(Y)$ is counted with multiplicity.

In our situation, when $Y = \mathbb{P}^k$, we have $\mathrm{volume}(g^n(Y)) \simeq d^{kn}$. Therefore, $h_t(f) \geq k \log d$. We can also deduce this inequality from Theorem 1.118 and the variational principle below.

End of the proof of Theorem 1.108. It remains to prove that $h_t(f) \leq k \log d$. Let Γ_n denote the graph of $(f, f^2, \dots, f^{n-1})$ in $(\mathbb{P}^k)^n$, i.e. the set of points

$$(z, f(z), f^2(z), \dots, f^{n-1}(z))$$

with z in $\mathbb{P}^k$. This is a manifold of dimension k. Let Π_i, $i = 0, \dots, n-1$, denote the projections from $(\mathbb{P}^k)^n$ onto the factors $\mathbb{P}^k$. We use on $(\mathbb{P}^k)^n$ the metric and the distance associated to the Kähler form $\omega_n := \sum \Pi_i^*(\omega_{\mathrm{FS}})$ induced by the Fubini-Study metrics ω_{FS} on the factors $\mathbb{P}^k$, see Appendix A.1. The following indicator $\mathrm{lov}(f)$ was introduced by Gromov, it measures the growth rate of the volume of Γ_n,

$$\mathrm{lov}(f) := \lim_{n \to \infty} \frac{1}{n} \log \mathrm{volume}(\Gamma_n).$$

The rest of the proof splits into two parts. We first show that the previous limit exists and is equal to $k \log d$ and then we prove the inequality $h_t(f) \leq \mathrm{lov}(f)$.

Using that $\Pi_0 : \Gamma_n \to \mathbb{P}^k$ is a bi-holomorphic map and that $f^i = \Pi_i \circ (\Pi_{0|\Gamma_n})^{-1}$, we obtain

$$k!\mathrm{volume}(\Gamma_n) = \int_{\Gamma_n} \omega_n^k = \sum_{0 \leq i_s \leq n-1} \int_{\Gamma_n} \Pi_{i_1}^*(\omega_{\mathrm{FS}}) \wedge \dots \wedge \Pi_{i_k}^*(\omega_{\mathrm{FS}})$$

$$= \sum_{0 \leq i_s \leq n-1} \int_{\mathbb{P}^k} (f^{i_1})^*(\omega_{\mathrm{FS}}) \wedge \dots \wedge (f^{i_k})^*(\omega_{\mathrm{FS}}).$$

The last sum contains n^k integrals that we can compute cohomologically. The above discussion on the action of f^n on cohomology implies that the last integral is equal to $d^{i_1 + \dots + i_k} \leq d^{kn}$. So, the sum is bounded from above by $n^k d^{kn}$. When $i_1 = \dots = i_k = n-1$, we see that $k!\mathrm{volume}(\Gamma_n) \geq d^{(n-1)k}$. Therefore, the limit in the definition of $\mathrm{lov}(f)$ exists and is equal to $k \log d$.

For the second step, we need the following classical estimate due to Lelong [LE], see also Appendix A.2.

Lemma 1.111 (Lelong). *Let A be an analytic set of pure dimension k in a ball B_r of radius r in $\mathbb{C}^N$. Assume that A contains the center of B_r. Then the $2k$-dimensional volume of A is at least equal to the volume of a ball of radius r in $\mathbb{C}^k$. In particular, we have*

$$\mathrm{volume}(A) \geq c_k r^{2k},$$

where $c_k > 0$ is a constant independent of N and of r.

We prove now the inequality $h_t(f) \leq \mathrm{lov}(f)$. Consider an (n, ε)-separated set $\mathscr{F}$ in $\mathbb{P}^k$. For each point $a \in \mathscr{F}$, let $a^{(n)}$ denote the corresponding point $(a, f(a), \ldots, f^{n-1}(a))$ in Γ_n and $B_{a,n}$ the ball of center $a^{(n)}$ and of radius $\varepsilon/2$ in $(\mathbb{P}^k)^n$. Since $\mathscr{F}$ is (n, ε)-separated, these balls are disjoint. On the other hand, by Lelong's inequality, $\mathrm{volume}(\Gamma_n \cap B_{a,n}) \geq c_k' \varepsilon^{2k}$, $c_k' > 0$. Note that Lelong's inequality is stated in the Euclidean metric. We can apply it using a fixed atlas of $\mathbb{P}^k$ and the corresponding product atlas of $(\mathbb{P}^k)^n$, the distortion is bounded. So, $\#\mathscr{F} \leq c_k'^{-1} \varepsilon^{-2k} \mathrm{volume}(\Gamma_n)$ and hence,

$$\frac{1}{n} \log \#\mathscr{F} \leq \frac{1}{n} \log(\mathrm{volume}(\Gamma_n)) + O\left(\frac{1}{n}\right).$$

It follows that $h_t(f) \leq \mathrm{lov}(f) = k \log d$. $\square$

We study the entropy of f on some subsets of $\mathbb{P}^k$. The following result is due to de Thélin and Dinh [DT3, D3].

Theorem 1.112. *Let f be a holomorphic endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$ and $\mathscr{J}_p$ its Julia set of order p, $1 \leq p \leq k$. If K is a subset of $\mathbb{P}^k$ such that $\overline{K} \cap \mathscr{J}_p = \varnothing$, then $h_t(f, K) \leq (p-1) \log d$.*

Proof. The proof is based on Gromov's idea as in Theorem 1.108 and on the speed of convergence towards the Green current. Recall that $\mathscr{J}_p$ is the support of the Green (p,p)-current T^p of f. Fix an open neighbourhood W of $\overline{K}$ such that $W \Subset \mathbb{P}^k \setminus \mathrm{supp}(T^p)$. Using the notations in Theorem 1.108, we only have to prove that

$$\mathrm{lov}(f, W) := \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{volume}(\Pi_0^{-1}(W) \cap \Gamma_n) \leq (p-1) \log d.$$

It is enough to show that $\mathrm{volume}(\Pi_0^{-1}(W) \cap \Gamma_n) \lesssim n^k d^{(p-1)n}$. As in Theorem 1.108, it is sufficient to check that for $0 \leq n_i \leq n$

$$\int_W (f^{n_1})^*(\omega_{\mathrm{FS}}) \wedge \ldots \wedge (f^{n_k})^*(\omega_{\mathrm{FS}}) \lesssim d^{(p-1)n}.$$

To this end, we prove by induction on (r, s), $0 \leq r \leq p$ and $0 \leq s \leq k - p + r$, that

$$\|T^{p-r} \wedge (f^{n_1})^*(\omega_{\mathrm{FS}}) \wedge \ldots \wedge (f^{n_s})^*(\omega_{\mathrm{FS}})\|_{W_{r,s}} \leq c_{r,s} d^{n(r-1)},$$

where $W_{r,s}$ is a neighbourhood of $\overline{W}$ and $c_{r,s} \geq 0$ is a constant independent of n and of n_i. We obtain the result by taking $r = p$ and $s = k$.

It is clear that the previous inequality holds when $r = 0$ and also when $s = 0$. In both cases, we can take $W_{r,s} = \mathbb{P}^k \setminus \mathrm{supp}(T^p)$ and $c_{r,s} = 1$. Assume the inequality for $(r-1, s-1)$ and $(r, s-1)$. Let $W_{r,s}$ be a neighbourhood of $\overline{W}$ strictly contained in $W_{r-1,s-1}$ and $W_{r,s-1}$. Let $\chi \geq 0$ be a smooth cut-off function with support in $W_{r-1,s-1} \cap W_{r,s-1}$ which is equal to 1 on $W_{r,s}$. We only have to prove that

$$\int T^{p-r} \wedge (f^{n_1})^*(\omega_{\mathrm{FS}}) \wedge \ldots \wedge (f^{n_s})^*(\omega_{\mathrm{FS}}) \wedge \chi \omega_{\mathrm{FS}}^{k-p+r-s} \leq c_{r,s} d^{n(r-1)}.$$

If g is the Green function of f, we have

$$(f^{n_1})^*(\omega_{\mathrm{FS}}) = d^{n_1}T - dd^c(g \circ f^{n_1}).$$

The above integral is equal to the sum of the following integrals

$$d^{n_1}\int T^{p-r+1} \wedge (f^{n_2})^*(\omega_{\mathrm{FS}}) \wedge \ldots \wedge (f^{n_s})^*(\omega_{\mathrm{FS}}) \wedge \chi \omega_{\mathrm{FS}}^{k-p+r-s}$$

and

$$-\int T^{p-r} \wedge dd^c(g \circ f^{n_1}) \wedge (f^{n_2})^*(\omega_{\mathrm{FS}}) \wedge \ldots \wedge (f^{n_s})^*(\omega_{\mathrm{FS}}) \wedge \chi \omega_{\mathrm{FS}}^{k-p+r-s}.$$

Using the case of $(r-1,s-1)$ we can bound the first integral by $cd^{n(r-1)}$. Stokes' theorem implies that the second integral is equal to

$$-\int T^{p-r} \wedge (f^{n_2})^*(\omega_{\mathrm{FS}}) \wedge \ldots \wedge (f^{n_s})^*(\omega_{\mathrm{FS}}) \wedge (g \circ f^{n_1}) dd^c \chi \wedge \omega_{\mathrm{FS}}^{k-p+r-s}$$

which is bounded by

$$\|g\|_\infty \|\chi\|_{\mathscr{C}^2} \|T^{p-r} \wedge (f^{n_2})^*(\omega_{\mathrm{FS}}) \wedge \ldots \wedge (f^{n_s})^*(\omega_{\mathrm{FS}})\|_{W_{r,s-1}}$$

since χ has support in $W_{r,s-1}$. We obtain the result using the $(r,s-1)$ case.   □

The above result suggests a local indicator of volume growth. Define for $a \in \mathbb{P}^k$

$$\mathrm{lov}(f,a) := \inf_{r>0} \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{volume}(\Pi_0^{-1}(B_r) \cap \Gamma_n),$$

where B_r is the ball of center a and of radius r. We can show that if $a \in \mathscr{I}_p \setminus \mathscr{I}_{p+1}$ and if $\overline{B}_r$ does not intersect $\mathscr{I}_{p+1}$, the above limsup is in fact a limit and is equal to $p\log d$. One can also consider the graph of f^n instead of Γ_n. The notion can be extended to meromorphic maps and its sub-level sets are analogues of Julia sets.

We discuss now the metric entropy, i.e. the entropy of an invariant measure, a notion due to Kolgomorov-Sinai. Let $g : X \to X$ be map on a space X which is measurable with respect to a σ-algebra $\mathscr{F}$. Let ν be an invariant probability measure for g. Let $\xi = \{\xi_1, \ldots, \xi_m\}$ be a measurable partition of X. The entropy of ν with respect to ξ is a measurement of the information we gain when we know that a point x belongs to a member of the partition generated by $g^{-i}(\xi)$ with $0 \le i \le n-1$.

The information we gain when we know that a point x belongs to ξ_i is a positive function $I(x)$ which depends only on $\nu(\xi_i)$, i.e. $I(x) = \varphi(\nu(\xi_i))$. The information given by independent events should be additive. In other words, we have

$$\varphi(\nu(\xi_i)\nu(\xi_j)) = \varphi(\nu(\xi_i)) + \varphi(\nu(\xi_j))$$

for $i \neq j$. Hence, $\varphi(t) = -c\log t$ with $c > 0$. With the normalization $c = 1$, the information function for the partition ξ is defined by

$$I_\xi(x) := \sum -\log \nu(\xi_i) \mathbf{1}_{\xi_i}(x).$$

The entropy of ξ is the average of I_ξ:

$$H(\xi) := \int I_\xi(x)d\nu(x) = -\sum \nu(\xi_i)\log \nu(\xi_i).$$

It is useful to observe that the function $t \mapsto -t\log t$ is concave on $]0,1]$ and has the maximal value e^{-1} at e^{-1}.

Consider now the information obtained if we measure the position of the orbit $x, g(x), \ldots, g^{n-1}(x)$ relatively to ξ. By definition, this is the measure of the entropy of the partition generated by $\xi, g^{-1}(\xi), \ldots, g^{-n+1}(\xi)$, which we denote by $\bigvee_{i=0}^{n-1} g^{-i}(\xi)$. The elements of this partition are $\xi_{i_1} \cap g^{-1}(\xi_{i_2}) \cap \ldots \cap g^{-n+1}(\xi_{i_{n-1}})$. It can be shown [W] that

$$h_\nu(g,\xi) := \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} g^{-i}(\xi)\right)$$

exists. *The entropy of the measure ν is defined as*

$$h_\nu(g) := \sup_\xi h_\nu(g,\xi).$$

Two measurable dynamical systems g on $(X,\mathscr{F},\nu)$ and g' on $(X',\mathscr{F}',\nu')$ are said to be *measurably conjugate* if there is a measurable invertible map $\pi : X \to X'$ such that $\pi \circ g = g' \circ \pi$ and $\pi_*(\nu) = \nu'$. In that case, we have $h_\nu(g) = h_{\nu'}(g')$. So, entropy is a conjugacy invariant. Note also that $h_\nu(g^n) = nh_\nu(g)$ and if g is invertible, $h_\nu(g^n) = |n|h_\nu(g)$ for $n \in \mathbb{Z}$. Moreover, if g is a continuous map of a compact metric space, then $\nu \mapsto h_\nu(g)$ is an affine function on the convex set of g-invariant probability measures [KH, p.164].

We say that a measurable partition ξ is *a generator* if up to sets of measure zero, $\mathscr{F}$ is the smallest σ-algebra containing ξ which is invariant under $g^n, n \in \mathbb{Z}$. A finite partition ξ is called *a strong generator* for a measure preserving dynamical system $(X,\mathscr{F},\nu,g)$ as above, if up to sets of zero ν-measure, $\mathscr{F}$ generated by $\bigcup_{n=0}^\infty g^{-n}(\xi)$. The following result of Kolmogorov-Sinai is useful in order to compute the entropy [W].

Theorem 1.113 (Kolmogorov-Sinai). *Let ξ be a strong generator for the dynamical system $(X,\mathscr{F},\nu,g)$ as above. Then*

$$h_\nu(g) = h_\nu(g,\xi).$$

We recall another useful theorem due to Brin-Katok [BK] which is valid for continuous maps $g : X \to X$ on a compact metric space. Let $B_n^g(x,\delta)$ denote the ball of center x and of radius δ with respect to the Bowen distance dist_n. We call $B_n^g(x,\delta)$ *the Bowen (n,δ)-ball*. Define local entropies of an invariant probability measure ν by

$$h_\nu(g,x) := \sup_{\delta > 0} \limsup_{n\to\infty} -\frac{1}{n}\log \nu(B_n^g(x,\delta))$$

and

$$h_\nu^-(g,x) := \sup_{\delta>0} \liminf_{n\to\infty} -\frac{1}{n} \log \nu(B_n^g(x,\delta)).$$

Theorem 1.114 (Brin-Katok). *Let $g : X \to X$ be a continuous map on a compact metric space. Let ν be an invariant probability measure of finite entropy. Then, $h_\nu(g,x) = h_\nu^-(g,x)$ and $h_\nu(g,g(x)) = h_\nu(g,x)$ for ν-almost every x. Moreover, $\langle \nu, h_\nu(g,\cdot)\rangle$ is equal to the entropy $h_\nu(g)$ of ν. In particular, if ν is ergodic, we have $h_\nu(g,x) = h_\nu(g)$ ν-almost everywhere.*

One can roughly say that $\nu(B_n^g(x,\delta))$ goes to zero at the exponential rate $e^{-h_\nu(g)}$ for δ small. We can deduce from the above theorem that if $Y \subset X$ is a Borel set with $\nu(Y) > 0$, then $h_t(g,Y) \geq h_\nu(g)$. The comparison with the topological entropy is given by the variational principle [KH, W].

Theorem 1.115 (variational principle). *Let $g : X \to X$ be a continuous map on a compact metric space. Then*

$$\sup h_\nu(g) = h_t(g),$$

where the supremum is taken over the invariant probability measures ν.

Newhouse proved in [NE] that if g is a smooth map on a smooth compact manifold, there is always a measure ν of maximal entropy, i.e. $h_\nu(g) = h_t(g)$. One of the natural question in dynamics is to find the measures which maximize entropy. Their supports are in some sense the most chaotic parts of the system. The notion of Jacobian of a measure is useful in order to estimate the metric entropy.

Let $g : X \to X$ be a measurable map as above which preserves a probability measure ν. Assume there is a countable partition (ξ_i) of X, such that the map g is injective on each ξ_i. *The Jacobian $J_\nu(g)$ of g with respect to ν is defined as the Radon-Nikodym derivative of $g^*(\nu)$ with respect to ν on each ξ_i.* Observe that $g^*(\nu)$ is well-defined on ξ_i since g restricted to ξ_i is injective. We have the following theorem due to Parry [P].

Theorem 1.116 (Parry). *Let g, ν be as above and $J_\nu(g)$ the Jacobian of g with respect to ν. Then*

$$h_\nu(g) \geq \int \log J_\nu(g) d\nu.$$

We now discuss the metric entropy of holomorphic maps on $\mathbb{P}^k$. The following result is a consequence of the variational principle and Theorems 1.108 and 1.112.

Corollary 1.117. *Let f be an endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$. Let ν be an invariant probability measure. Then $h_\nu(f) \leq k \log d$. If the support of ν does not intersect the Julia set $\mathscr{J}_p$ of order p, then $h_\nu(f) \leq (p-1)\log d$. In particular, if ν is ergodic and $h_\nu(f) > (p-1)\log d$, then ν is supported on $\mathscr{J}_p$.*

In the following result, the value of the metric entropy was obtained in [BD2,S3] and the uniqueness was obtained by Briend-Duval in [BD2]. The case of dimension 1 is due to Freire-Lopès-Mañé [FL] and Lyubich [LY].

Theorem 1.118. *Let f be an endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$. Then the equilibrium measure $\quad$ of f is the unique invariant measure of maximal entropy $k \log d$.*

Proof. We have seen in Corollary 1.117 that $h \ (f) \leq k \log d$. Moreover, $\quad$ has no mass on analytic sets, in particular on the critical set of f. Therefore, if f is injective on a Borel set K, then $f_*(\mathbf{1}_K) = \mathbf{1}_{f(K)}$ and the total invariance of $\quad$ implies that $\quad (f(K)) = d^k \ (K)$. So, $\quad$ is a measure of constant Jacobian d^k. It follows from Theorem 1.116 that its entropy is at least equal to $k \log d$. So, $h \ (f) = k \log d$.

Assume now that there is another invariant probability measure ν of entropy $k \log d$. We are looking for a contradiction. Since entropy is an affine function on ν, we can assume that ν is ergodic. This measure has no mass on proper analytic sets of $\mathbb{P}^k$ since otherwise its entropy is at most equal to $(k-1) \log d$, see Exercise 1.122 below. By Theorem 1.45, ν is not totally invariant, so it is not of constant Jacobian. Since $\quad$ has no mass on critical values of f, there is a simply connected open set U, not necessarily connected, such that $f^{-1}(U)$ is a union $U_1 \cup \ldots \cup U_{d^k}$ of disjoint open sets such that $f : U_i \to U$ is bi-holomorphic. One can choose U and U_i such that the U_i do not have the same ν-measure, otherwise $\quad = \nu$. So, we can assume that $\nu(U_1) > d^{-k}$. This is possible since two ergodic measures are mutually singular. Here, it is necessarily to chose U so that $\quad (\mathbb{P}^k \setminus U)$ is small.

Choose an open set $W \Subset U_1$ such that $\nu(W) > \sigma$ for some constant $\sigma > d^{-k}$. Let m be a fixed integer and let Y be the set of points x such that for every $n \geq m$, there are at least $n\sigma$ points $f^i(x)$ with $0 \leq i \leq n-1$ which belong to W. If m is large enough, Birkhoff's theorem implies that Y has positive ν-measure. By Brin-Katok's theorem 1.114, we have $h_t(f, Y) \geq h_\nu(f) = k \log d$.

Consider the open sets $\mathcal{U}_\alpha := U_{\alpha_0} \times \cdots \times U_{\alpha_{n-1}}$ in $(\mathbb{P}^k)^n$ such that there are at least $n\sigma$ indices α_i equal to 1. A straighforward computation shows that the number of such open sets is $\leq d^{k\rho n}$ for some constant $\rho < 1$. Let $\mathcal{V}_n$ denote the union of these $\mathcal{U}_\alpha$. Using the same arguments as in Theorem 1.108, we get that

$$k \log d \leq h_t(f, Y) \leq \lim_{n \to \infty} \frac{1}{n} \log \text{volume}(\Gamma_n \cap \mathcal{V}_n)$$

and

$$k! \text{volume}(\Gamma_n \cap \mathcal{V}_n) = \sum_{0 \leq i_s \leq n-1} \sum_\alpha \int_{\Gamma_n \cap \mathcal{U}_\alpha} \Pi_{i_1}^*(\omega_{\text{FS}}) \wedge \ldots \wedge \Pi_{i_k}^*(\omega_{\text{FS}}).$$

Fix a constant λ such that $\rho < \lambda < 1$. Let I denote the set of multi-indices $i = (i_1, \ldots, i_k)$ in $\{0, \ldots, n-1\}^k$ such that $i_s \geq n\lambda$ for every s. We distinguish two cases where $i \notin I$ or $i \in I$. In the first case, we have

$$\sum_\alpha \int_{\Gamma_n \cap \mathcal{U}_\alpha} \Pi_{i_1}^*(\omega_{\text{FS}}) \wedge \ldots \wedge \Pi_{i_k}^*(\omega_{\text{FS}}) \leq \int_{\Gamma_n} \Pi_{i_1}^*(\omega_{\text{FS}}) \wedge \ldots \wedge \Pi_{i_k}^*(\omega_{\text{FS}})$$

$$= \int_{\mathbb{P}^k} (f^{i_1})^*(\omega_{\text{FS}}) \wedge \ldots \wedge (f^{i_k})^*(\omega_{\text{FS}})$$

$$= d^{i_1 + \cdots + i_k} \leq d^{(k-1+\lambda)n},$$

since $i_1 + \cdots + i_k \leq (k-1+\lambda)n$.

Consider the second case with multi-indices $i \in I$. Let q denote the integer part of λn and W_α the projection of $\Gamma_n \cap \mathcal{U}_\alpha$ on $\mathbb{P}^k$ by Π_0. Observe that the choice of the open sets U_i implies that f^q is injective on W_α. Therefore,

$$
\begin{aligned}
\sum_\alpha \int_{\Gamma_n \cap \mathcal{U}_\alpha} & \Pi_{i_1}^*(\omega_{\mathrm{FS}}) \wedge \ldots \wedge \Pi_{i_k}^*(\omega_{\mathrm{FS}}) \\
&\leq \sum_\alpha \int_{W_\alpha} (f^{i_1})^*(\omega_{\mathrm{FS}}) \wedge \ldots \wedge (f^{i_k})^*(\omega_{\mathrm{FS}}) \\
&= \sum_\alpha \int_{W_\alpha} (f^q)^* \left[(f^{i_1-q})^*(\omega_{\mathrm{FS}}) \wedge \ldots \wedge (f^{i_k-q})^*(\omega_{\mathrm{FS}}) \right] \\
&\leq \sum_\alpha \int_{\mathbb{P}^k} (f^{i_1-q})^*(\omega_{\mathrm{FS}}) \wedge \ldots \wedge (f^{i_k-q})^*(\omega_{\mathrm{FS}}).
\end{aligned}
$$

Recall that the number of open sets $\mathcal{U}_\alpha$ is bounded by $d^{k\rho n}$. So, the last sum is bounded by

$$
d^{k\rho n} d^{(i_1-q)+\cdots+(i_k-q)} \leq d^{k\rho n} d^{k(n-q)} \lesssim d^{k(1+\rho-\lambda)n}.
$$

Finally, since the number of multi-indices i is less than n^k, we deduce from the above estimates that

$$
k! \mathrm{volume}(\Gamma_n \cap \mathcal{V}_n) \lesssim n^k d^{(k-1+\lambda)n} + n^k d^{k(1+\rho-\lambda)n}.
$$

This contradicts the above bound from below of $\mathrm{volume}(\Gamma_n \cap \mathcal{V}_n)$. $\square$

The remaining part of this paragraph deals with Lyapounov exponents associated to the measure   and their relations with the Hausdorff dimension of   . Results in this direction give some information about the rough geometrical behaviour of the dynamical system on the most chaotic locus. An abstract theory was developed by Oseledec and Pesin, see e.g. [KH]. However, it is often difficult to show that a given dynamical system has non-vanishing Lyapounov exponents. In complex dynamics as we will see, the use of holomorphicity makes the goal reachable. We first introduce few notions.

Let A be a linear endomorphism of $\mathbb{R}^k$. We can write $\mathbb{R}^k$ as the direct sum $\oplus E_i$ of invariant subspaces on which all the complex eigenvalues of A have the same modulus. This decomposition of $\mathbb{R}^k$ describes clearly the geometrical behaviour of the dynamical system associated to A. An important part in the dynamical study with respect to an invariant measure is to describe geometrical aspects following the directional dilation or contraction indicators.

Consider a smooth dynamical system $g : X \to X$ and an invariant ergodic probability measure ν. The map g induces a linear map from the tangent space at x to the tangent space at $g(x)$. This linear map is given by a square matrix when we fix local coordinates near x and $g(x)$. Then, we obtain a function on X with values in $\mathrm{GL}(\mathbb{R}, k)$ where k denotes the real dimension of X. We will study the sequence of such functions associated to the sequence of iterates (g^n) of g.

Consider a more abstract setting. Let $g : X \to X$ be a measurable map and ν an invariant probability measure. Let $A : X \to \mathrm{GL}(\mathbb{R}, k)$ be a measurable function. Define for $n \geq 0$

$$A_n(x) := A(g^{n-1}(x)) \ldots A(x).$$

These functions satisfy the identity

$$A_{n+m}(x) = A_n(g^m(x)) A_m(x)$$

for $n, m \geq 0$. We say that the sequence (A_n) is *the multiplicative cocycle* over X generated by A.

The following Oseledec's multiplicative ergodic theorem is related to the Kingman's sub-multiplicative ergodic theorem [KH, W]. It can be seen as a generalization of the above property of a single square matrix A.

Theorem 1.119 (Oseledec). *Let $g : X \to X$, ν and the cocycle (A_n) be as above. Assume that ν is ergodic and that $\log^+ \|A^{\pm 1}(x)\|$ are in $L^1(\nu)$. Then there is an integer m, real numbers $\chi_1 < \cdots < \chi_m$, and for ν-almost every x, a unique decomposition of $\mathbb{R}^k$ into a direct sum of linear subspaces*

$$\mathbb{R}^k = \bigoplus_{i=1}^{m} E_i(x)$$

such that

1. *The dimension of $E_i(x)$ does not depend on x.*
2. *The decomposition is invariant, that is, $A(x)$ sends $E_i(x)$ to $E_i(g(x))$.*
3. *We have locally uniformly on vectors v in $E_i(x) \setminus \{0\}$*

$$\lim_{n \to \infty} \frac{1}{n} \log \|A_n(x) \cdot v\| = \chi_i.$$

4. *For $S \subset \{1, \ldots, m\}$, define $E_S(x) := \oplus_{i \in S} E_i(x)$. If S, S' are disjoint, then the angle between $E_S(x)$ and $E_{S'}(x)$ is a tempered function, that is,*

$$\lim_{n \to \infty} \frac{1}{n} \log \sin \left| \angle \big(E_S(g^n(x)), E_{S'}(g^n(x)) \big) \right| = 0.$$

The result is still valid for non-ergodic systems but the constants m and χ_i should be replaced with invariant functions. If g is invertible, the previous decomposition is the same for g^{-1} where the exponents χ_i are replaced with $-\chi_i$. The result is also valid in the complex setting where we replace $\mathbb{R}$ with $\mathbb{C}$ and $\mathrm{GL}(\mathbb{R}, k)$ by $\mathrm{GL}(\mathbb{C}, k)$. In this case, the subspaces $E_i(x)$ are complex.

We now come back to a smooth dynamical system $g : X \to X$ on a compact manifold. We assume that the Jacobian $J(g)$ of g associated to a smooth volume form satisfies $\langle \nu, \log J(g) \rangle > -\infty$. Under this hypothesis, we can apply Oseledec's theorem to the cocycle induced by g on the tangent bundle of X; this allows to decompose, ν-almost everywhere, the tangent bundle into invariant sub-bundles.

The corresponding constants χ_i are called *Lyapounov exponents* of g with respect to ν. The dimension of E_i is *the multiplicity* of χ_i. These notions do not depend on the choice of local coordinates on X. The Lyapounov exponents of g^n are equal to $n\chi_i$. We say that the measure ν is *hyperbolic* if no Lyapounov exponent is zero. It is not difficult to deduce from the Oseledec's theorem that the sum of Lyapounov exponents of ν is equal to $\langle \nu, \log J(g) \rangle$. The reader will find in [KH] a theorem due to Pesin, called *the ε-reduction theorem*, which generalizes Theorem 1.119. It gives some coordinate changes on $\mathbb{R}^k$ which allow to write $A(x)$ in the form of a diagonal block matrix with explicit estimates on the distortion.

The following result due to Briend-Duval [BD1], shows that endomorphisms in $\mathbb{P}^k$ are expansive with respect to the equilibrium measures. We give here a new proof using Proposition 1.51. Note that there are k Lyapounov exponents counted with multiplicity. If we consider these endomorphism as real maps, we should count twice the Lyapounov exponents.

Theorem 1.120. *Let f be a holomorphic endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$. Then the equilibrium measure of f is hyperbolic. More precisely, its Lyapounov exponents are larger or equal to $\frac{1}{2}\log d$.*

Proof. Since the measure is PB, quasi-p.s.h. functions are -integrable. It is not difficult to check that if $J(f)$ is the Jacobian of f with respect to the Fubini-Study metric, then $\log J(f)$ is a quasi-p.s.h. function. Therefore, we can apply Oseledec's theorem 1.119. We deduce from this result that the smallest Lyapounov exponent of is equal to

$$\chi := \lim_{n \to \infty} -\frac{1}{n}\log \|Df^n(x)^{-1}\|$$

for -almost every x. By Proposition 1.51, there is a ball B of positive measure which admits at least $\frac{1}{2}d^{kn}$ inverse branches $g_i : B \to U_i$ for f^n with U_i of diameter $\leq d^{-n/2}$. If we slightly reduce the ball B, we can assume that $\|Dg_i\| \leq Ad^{-n/2}$ for some constant $A > 0$. This is a simple consequence of Cauchy's formula. It follows that $\|(Df^n)^{-1}\| \leq Ad^{-n/2}$ on U_i. The union V_n of the U_i is of measure at least equal to $\frac{1}{2}(B)$. Therefore, by Fatou's lemma,

$$\frac{1}{2}(B) \leq \limsup_{n\to\infty}\langle\ ,\mathbf{1}_{V_n}\rangle \leq \langle\ ,\limsup\mathbf{1}_{V_n}\rangle = \langle\ ,\mathbf{1}_{\limsup V_n}\rangle.$$

Hence, there is a set $K := \limsup V_n$ of positive measure such that if x is in K, we have $\|Df^n(x)^{-1}\| \leq Ad^{-n/2}$ for infinitely many of n. The result follows. $\square$

Note that in a recent work [DT2], de Thélin proved that if ν is an invariant measure of entropy strictly larger than $(k-1)\log d$ such that the logarithm of the Jacobian is integrable, then the associated Lyapounov exponents are strictly positive. He also obtained for these exponents some explicit estimates from below. His method is more powerful and is valid in a very general setting.

The Hausdorff dimension $\dim_H(\nu)$ of a probability measure ν on $\mathbb{P}^k$ is the infimum of the numbers $\alpha \geq 0$ such that there is a Borel set K of Hausdorff dimension α of full measure, i.e. $\nu(K) = 1$. Hausdorff dimension says how the

measure fills out its support. The following result was obtained by Binder-DeMarco [BI] and Dinh-Dupont [DD]. The fact that   has positive dimension has been proved in [S3]; indeed, a lower bound is given in terms of the Hölder continuity exponent of the Green function g.

Theorem 1.121. *Let* f *be an endomorphism of algebraic degree* $d \geq 2$ *of* $\mathbb{P}^k$ *and   its equilibrium measure. Let* $\chi_1, \ldots, \chi_k$ *denote the Lyapounov exponents of   ordered by* $\chi_1 \geq \cdots \geq \chi_k$ *and* Σ *their sum. Then*

$$\frac{k \log d}{\chi_1} \leq \dim_H(\ ) \leq 2k - \frac{2\Sigma - k \log d}{\chi_1}.$$

The proof is quite technical. It is based on a delicate study of the inverse branches of balls along a generic negative orbit. We will not give the proof here. A better estimate in dimension 2, probably the sharp one, was recently obtained by Dupont. Indeed, Binder-DeMarco conjecture that the Hausdorff dimension of   satisfies

$$\dim_H(\ ) = \frac{\log d}{\chi_1} + \cdots + \frac{\log d}{\chi_k}.$$

Dupont gives in [DP3] results in this direction.

Exercise 1.122. Let X be an analytic subvariety of pure dimension p in $\mathbb{P}^k$. Let f be an endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$. Show that $h_t(f, X) \leq p \log d$.

Exercise 1.123. Let $f : X \to X$ be a smooth map and K an invariant compact subset of X. Assume that K is hyperbolic, i.e. there is a continuously varying decomposition $TX_{|K} = E \oplus F$ of the tangent bundle of X restricted to K, into the sum of two invariant vector bundles such that $\|Df\| < 1$ on E and $\|(Df)^{-1}\| < 1$ on F for some smooth metric near K. Show that f admits a hyperbolic ergodic invariant measure supported on K.

Exercise 1.124. Let $f : X \to X$ be a holomorphic map on a compact complex manifold and let ν be an ergodic invariant measure. Show that in Theorem 1.119 applied to the action of f on the complex tangent bundle, the spaces $E_i(x)$ are complex.

Exercise 1.125. Let $\alpha > 0$ be a constant. Show that there is an endomorphism f of $\mathbb{P}^k$ such that the Hausdorff dimension of the equilibrium measure of f is smaller than α. Show that there is an endomorphism f such that its Green function g is not α-Hölder continuous.

Notes. We do not give here results on local dynamics near a fixed point. If this point is non-critical attractive or repelling, a theorem of Poincaré says that the map is locally conjugated to a polynomial map [ST]. Maps which are tangent to the identity or are semi-attractive at a fixed point, were studied by Abate and Hakim [A, H1, H2, H3], see also Abate, Bracci and Tovena [ABT]. Dynamics near a super-attractive fixed point in dimension $k = 2$ was studied by Favre-Jonsson using a theory of valuations in [FJ1].

The study of the dynamical system outside the support of the equilibrium measure is not yet developed. Some results on attracting sets, attracting currents, etc. were obtained by de Thélin, Dinh, Fornæss, Jonsson, Sibony, Weickert [DT1, DT2, D3, FS6, FW, JW], see also Bonifant, Dabija, Milnor and Taflin [BO, T], Mihailescu and Urbański [MI, MIH].

In dimension 1, Fatou and Julia considered their theory as an investigation to solve some functional equations. In particular, they found all the commuting pairs of polynomials [F, JU], see also Ritt [R] and Eremenko [E] for the case of rational maps. Commuting endomorphisms of $\mathbb{P}^k$ were studied by the authors in [DS0]. A large family of solutions are Lattès maps. We refer to Berteloot, Dinh, Dupont, Loeb, Molino [BR, BDM, BL, D4, DP1] for a study of this class of maps, see also Milnor [MIL] for the dimension 1 case.

We do not consider here bifurcation problems for families of maps and refer to Bassanelli-Berteloot [BB] and Pham [PH] for this subject. Some results will be presented in the next chapter.

In [Z], Zhang considers some links between complex dynamics and arithmetic questions. He is interested in polarized holomorphic maps on Kähler varieties, i.e. maps which multiply a Kähler class by an integer. If the Kähler class is integral, the variety can be embedded into a projective space $\mathbb{P}^k$ and the maps extend to endomorphisms of $\mathbb{P}^k$. So, several results stated above can be directely applied to that situation. In general, most of the results for endomorphisms in $\mathbb{P}^k$ can be easily extended to general polarized maps. In the unpublished preprint [DS14], the authors considered the situation of smooth compact Kähler manifolds. We recall here the main result.

Let (X, ω) be an arbitrary compact Kähler manifold of dimension k. Let f be a holomorphic endomorphism of X. We assume that f is open. The spectral radius of f^* acting on $H^{p,p}(X, \mathbb{C})$ is called *the dynamical degree of order p* of f. It can be computed by the formula

$$d_p := \lim_{n\to\infty} \left(\int_X (f^n)^*(\omega^p) \wedge \omega^{k-p} \right)^{1/n}.$$

The last degree d_k is the topological degree of f, i.e. equal to the number of points in a generic fiber of f. We also denote it by d_t.

Assume that $d_t > d_p$ for $1 \leq p \leq k-1$. Then, there is a maximal proper analytic subset $\mathcal{E}$ of X which is totally invariant by f, i.e. $f^{-1}(\mathcal{E}) = f(\mathcal{E}) = \mathcal{E}$. If δ_a is a Dirac mass at $a \notin \mathcal{E}$, then $d_t^{-n}(f^n)^*(\delta_a)$ converge to a probability measure , which does not depend on a. This is *the equilibrium measure* of f. It satisfies $f^*( ) = d_t$ and $f_*( ) = $. If J is the Jacobian of f with respect to ω^k then $\langle , \log J \rangle \geq \log d_t$. The measure   is K-mixing and hyperbolic with Lyapounov exponents larger or equal to $\frac{1}{2} \log(d_t/d_{k-1})$. Moreover, there are sets $\mathscr{P}_n$ of repelling periodic points of order n, on supp( ) such that the probability measures equidistributed on $\mathscr{P}_n$ converge to , as n goes to infinity. If the periodic points of period n are isolated for every n, an estimate on the norm of $(f^n)^*$ on $H^{p,q}(X, \mathbb{C})$ obtained in [D22], implies that the number of these periodic points is $d_t^n + o(d_t^n)$. Therefore, periodic points are equidistributed with respect to . We can prove without difficulty that   is the unique invariant measure of maximal entropy $\log d_t$ and is moderate. Then, we can extend the stochastic properties obtained for $\mathbb{P}^k$ to this more general setting.

When f is polarized by the cohomology class $[\omega]$ of a Kähler form ω, there is a constant $\lambda \geq 1$ such that $f^*[\omega] = \lambda[\omega]$. It is not difficult to check that $d_p = \lambda^p$. The above results can be applied for such a map when $\lambda > 1$. In which case, periodic points of a given period are isolated. Note also that Theorem 1.108 can be extended to this case.

2 Polynomial-like Maps in Higher Dimension

In this section we consider a large family of holomorphic maps in a semi-local setting: the polynomial-like maps. They can appear as a basic block in the study of some meromorphic maps on compact manifolds. The main reference for this section is our article [DS1] where the dd^c-method in dynamics was introduced. Endomorphisms of $\mathbb{P}^k$ can be considered as a special case of polynomial-like maps. However, in general, there is no Green $(1,1)$-current for such maps. The notion of dynamical degrees for polynomial-like maps replaces the algebraic degree. Under

natural assumptions on dynamical degrees, we prove that the measure of maximal entropy is non-uniformly hyperbolic and we study its sharp ergodic properties.

2.1 Examples, Degrees and Entropy

Let V be a convex open set in $\mathbb{C}^k$ and $U \Subset V$ an open subset. A proper holomorphic map $f : U \to V$ is called *a polynomial-like map*. Recall that a map $f : U \to V$ is proper if $f^{-1}(K) \Subset U$ for every compact subset K of V. The map f sends the boundary of U to the boundary of V; more precisely, the points near ∂U are sent to points near ∂V. So, polynomial-like maps are somehow expansive in all directions, but the expansion is in the geometrical sense. In general, they may have a non-empty critical set. A polynomial-like mapping $f : U \to V$ defines a ramified covering over V. The degree d_t of this covering is also called *the topological degree*. It is equal to the number of points in a generic fiber, or in any fiber if we count the multiplicity.

Polynomial-like maps are characterized by the property that their graph Γ in $U \times V$ is in fact a submanifold of $V \times V$, that is, Γ is closed in $V \times V$. So, any small perturbation of f is polynomial-like of the same topological degree d_t, provided that we reduce slightly the open set V. We will construct large families of polynomial-like maps. In dimension one, it was proved by Douady-Hubbard [DH] that such a map is conjugated to a polynomial via a Hölder continuous homeomorphism. Many dynamical properties can be deduced from the corresponding properties of polynomials. In higher dimension, the analogous statement is not valid. Some new dynamical phenomena appear for polynomial-like mappings, that do not exist for polynomial maps. We use here an approach completely different from the one dimensional case, where the basic tool is the Riemann measurable mapping theorem.

Let $f : \mathbb{C}^k \to \mathbb{C}^k$ be a holomorphic map such that the hyperplane at infinity is attracting in the sense that $\|f(z)\| \geq A\|z\|$ for some constant $A > 1$ and for $\|z\|$ large enough. If V is a large ball centered at 0, then $U := f^{-1}(V)$ is strictly contained in V. Therefore, $f : U \to V$ is a polynomial-like map. Small transcendental perturbations of f, as we mentioned above, give a large family of polynomial-like maps. Observe also that the dynamical study of holomorphic endomorphisms on $\mathbb{P}^k$ can be reduced to polynomial-like maps by lifting to a large ball in $\mathbb{C}^{k+1}$. We give now other explicit examples.

Example 2.1. Let $f = (f_1, \ldots, f_k)$ be a polynomial map in $\mathbb{C}^k$, with $\deg f_i = d_i \geq 2$. Using a conjugacy with a permutation of coordinates, we can assume that $d_1 \geq \cdots \geq d_k$. Let f_i^+ denote the homogeneous polynomial of highest degree in f_i. If $\{f_1^+ = \cdots = f_k^+ = 0\}$ is reduced to $\{0\}$, then f is polynomial-like in any large ball of center 0. Indeed, define $d := d_1 \ldots d_k$ and $\pi(z_1, \ldots, z_k) := (z_1^{d/d_1}, \ldots, z_k^{d/d_k})$. Then, $\pi \circ f$ is a polynomial map of algebraic degree d which extends holomorphically at infinity to an endomorphism of $\mathbb{P}^k$. Therefore, $\|\pi \circ f(z)\| \gtrsim \|z\|^d$ for $\|z\|$ large enough. The estimate $\|f(z)\| \gtrsim \|z\|^{d_k}$ near infinity follows easily. If we consider the extension of f to $\mathbb{P}^k$, we obtain in general a meromorphic map which is not holomorphic. Small pertubations f_ε of f may have indeterminacy points in $\mathbb{C}^k$ and a priori, indeterminacy points of the sequence $(f_\varepsilon^n)_{n \geq 1}$ may be dense in $\mathbb{P}^k$.

Example 2.2. The map $(z_1, z_2) \mapsto (z_1^2 + az_2, z_1)$, $a \neq 0$, is not polynomial-like. It is invertible and the point $[0:0:1]$ at infinity, in homogeneous coordinates $[z_0:z_1:z_2]$, is an attractive fixed point for f^{-1}. Hence, the set $\mathscr{K}$ of points $z \in \mathbb{C}^2$ with bounded orbit, clusters at $[0:0:1]$.

The map $f(z_1, z_2) := (z_2^d, 2z_1)$, $d \geq 2$, is polynomial-like in any large ball of center 0. Considered as a map on $\mathbb{P}^2$, it is only meromorphic with an indeterminacy point $[0:1:0]$. On a fixed large ball of center 0, the perturbed maps $f_\varepsilon := (z_2^d + \varepsilon e^{z_1}, 2z_1 + \varepsilon e^{z_2})$ are polynomial-like, in an appropriate open set U.

Consider a general polynomial-like map $f : U \to V$ of topological degree $d_t \geq 2$. We introduce several growth indicators of the action of f on forms or currents. Define $f^n := f \circ \cdots \circ f$, n times, *the iterate of order n of f*. This map is only defined on $U_{-n} := f^{-n}(V)$. The sequence (U_{-n}) is decreasing: we have $U_{-n-1} = f^{-1}(U_{-n}) \Subset U_{-n}$. Their intersection $\mathscr{K} := \cap_{n \geq 0} U_{-n}$ is a non-empty compact set that we call *the filled Julia set* of f. The filled Julia set is totally invariant: we have $f^{-1}(\mathscr{K}) = \mathscr{K}$ which implies that $f(\mathscr{K}) = \mathscr{K}$. Only for x in $\mathscr{K}$, the infinite orbit $x, f(x), f^2(x), \ldots$ is well-defined. The preimages $f^{-n}(x)$ by f^n are defined for every $n \geq 0$ and every x in V.

Let $\omega := dd^c \|z\|^2$ denote the standard Kähler form on $\mathbb{C}^k$. Recall that the mass of a positive (p, p)-current S on a Borel set K is given by $\|S\|_K := \int_K S \wedge \omega^{k-p}$. Define the *dynamical degree of order p* of f, for $0 \leq p \leq k$, by

$$d_p(f) := \limsup_{n \to \infty} \|(f^n)_*(\omega^{k-p})\|_W^{1/n} = \limsup_{n \to \infty} \|(f^n)^*(\omega^p)\|_{f^{-n}(W)}^{1/n},$$

where $W \Subset V$ is a neighbourhood of $\mathscr{K}$. For simplicity, when there is no confusion, this degree is also denoted by d_p. We have the following lemma.

Lemma 2.3. *The degrees d_p do not depend on the choice of W. Moreover, we have $d_0 \leq 1$, $d_k = d_t$ and the dynamical degree of order p of f^m is equal to d_p^m.*

Proof. Let $W' \subset W$ be another neighbourhood of $\mathscr{K}$. For the first assertion, we only have to show that

$$\limsup_{n \to \infty} \|(f^n)^*(\omega^p)\|_{f^{-n}(W)}^{1/n} \leq \limsup_{n \to \infty} \|(f^n)^*(\omega^p)\|_{f^{-n}(W')}^{1/n}.$$

By definition of $\mathscr{K}$, there is an integer N such that $f^{-N}(V) \Subset W'$. Since $(f^N)^*(\omega^p)$ is smooth on $f^{-N}(V)$, we can find a constant $A > 0$ such that $(f^N)^*(\omega^p) \leq A\omega^p$ on $f^{-N}(W)$. We have

$$\limsup_{n \to \infty} \|(f^n)^*(\omega^p)\|_{f^{-n}(W)}^{1/n} = \limsup_{n \to \infty} \|(f^{n-N})^*((f^N)^*(\omega^p))\|_{f^{-n+N}(f^{-N}(W))}^{1/n}$$

$$\leq \limsup_{n \to \infty} \|A(f^{n-N})^*(\omega^p)\|_{f^{-n+N}(W')}^{1/n}$$

$$= \limsup_{n \to \infty} \|(f^n)^*(\omega^p)\|_{f^{-n}(W')}^{1/n}.$$

This proves the first assertion.

It is clear from the definition that $d_0 \leq 1$. Since f has topological degree d_t the pull-back of a positive measure multiplies the mass by d_t. Therefore, $d_k = d_t$. For the last assertion of the lemma, we only have to check that

$$\limsup_{n \to \infty} \|(f^n)^*(\omega^p)\|_{f^{-n}(W)}^{1/n} \leq \limsup_{s \to \infty} \|(f^{ms})^*(\omega^p)\|_{f^{-ms}(W)}^{1/ms}.$$

To this end, we proceed as above. Write $n = ms + r$ with $0 \leq r \leq m - 1$. We obtain the result using that $(f^r)^*(\omega^p) \leq A\omega^p$ on a fixed neighbourhood of $\mathscr{K}$ for $0 \leq r \leq m - 1$. $\qquad\square$

The main result of this paragraph is the following formula for the entropy.

Theorem 2.4. *Let $f : U \to V$ be a polynomial-like map of topological degree $d_t \geq 2$. Let $\mathscr{K}$ be the filled Julia set of f. Then, the topological entropy of f on $\mathscr{K}$ is equal to $h_t(f, \mathscr{K}) = \log d_t$. Moreover, all the dynamical degrees d_p of f are smaller or equal to d_t.*

We need the following lemma where we use standard metrics on Euclidean spaces.

Lemma 2.5. *Let V be an open set of $\mathbb{C}^k$, U a relatively compact subset of V and L a compact subset of $\mathbb{C}$. Let π denote the canonical projection from $\mathbb{C}^m \times V$ onto V. Suppose Γ is an analytic subset of pure dimension k of $\mathbb{C}^m \times V$ contained in $L^m \times V$. Assume also that $\pi : \Gamma \to V$ defines a ramified covering of degree d_Γ. Then, there exist constants $c > 0$, $s > 0$, independent of Γ and m, such that*

$$\mathrm{volume}(\Gamma \cap \mathbb{C}^m \times U) \leq cm^s d_\Gamma.$$

Proof. Since the problem is local on V, we can assume that V is the unit ball of $\mathbb{C}^k$ and U is the closed ball of center 0 and of radius $1/2$. We can also assume that L is the closed unit disc in $\mathbb{C}$. Denote by $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_k)$ the coordinates on $\mathbb{C}^m$ and on $\mathbb{C}^k$. Let ε be a $k \times m$ matrix whose entries have modulus bounded by $1/8mk$. Define $\pi_\varepsilon(x,y) := y + \varepsilon x$, $\Gamma_\varepsilon := \Gamma \cap \{\|\pi_\varepsilon\| < 3/4\}$ and $\Gamma^* := \Gamma \cap (L^m \times U)$.

We first show that $\Gamma^* \subset \Gamma_\varepsilon$. Consider a point $(x, y) \in \Gamma^*$. We have $|x_i| < 1$ and $\|y\| \leq 1/2$. Hence,
$$\|\pi_\varepsilon(x, y)\| \leq \|y\| + \|\varepsilon x\| < 3/4.$$
This implies that $(x, y) \in \Gamma_\varepsilon$.

Now, we prove that for every $a \in \mathbb{C}^k$ with $\|a\| < 3/4$, we have $\#\pi_\varepsilon^{-1}(a) \cap \Gamma = d_\Gamma$, where we count the multiplicities of points. To this end, we show that $\#\pi_{t\varepsilon}^{-1}(a) \cap \Gamma$ does not depend on $t \in [0, 1]$. So, it is sufficient to check that the union of the sets $\pi_{t\varepsilon}^{-1}(a) \cap \Gamma$ is contained in the compact subset $\Gamma \cap \{\|\pi\| \leq 7/8\}$ of Γ. Let $(x, y) \in \Gamma$ and $t \in [0, 1]$ such that $\pi_{t\varepsilon}(x, y) = a$. We have

$$3/4 > \|a\| = \|\pi_{t\varepsilon}(x, y)\| \geq \|y\| - t\|\varepsilon x\|.$$

It follows that $\|y\| < 7/8$ and hence $(x, y) \in \Gamma \cap \{\|\pi\| \leq 7/8\}$.

Let B denote the ball of center 0 and of radius $3/4$ in $\mathbb{C}^k$. We have for some constant $c' > 0$

$$\int_{\Gamma^*} \pi_{\varepsilon}^*(\omega^k) \leq \int_{\Gamma_{\varepsilon}} \pi_{\varepsilon}^*(\omega^k) = d_{\Gamma} \int_B \omega^k = c' d_{\Gamma}.$$

Let $\Theta := dd^c\|x\|^2 + dd^c\|y\|^2$ be the standard Kähler $(1,1)$-form on $\mathbb{C}^m \times \mathbb{C}^k$. We have

$$\text{volume}(\Gamma \cap \mathbb{C}^m \times U) = \int_{\Gamma \cap \mathbb{C}^m \times U} \Theta^k.$$

It suffices to bound Θ by a linear combination of $2m+1$ forms of type $\pi_{\varepsilon}^*(\omega)$ with coefficients of order $\simeq m^2$ and then to use the previous estimates. Recall that $\omega = dd^c\|y\|^2$. So, we only have to bound $\sqrt{-1}dx_i \wedge d\bar{x}_i$ by a combination of $(1,1)$-forms of type $\pi_{\varepsilon}^*(\omega)$. Consider $\delta := 1/8mk$ and $\pi_{\varepsilon}(x,y) := (y_1 + \delta x_i, y_2, \ldots, y_k)$. We have

$$\sqrt{-1}dx_i \wedge d\bar{x}_i = \frac{4\sqrt{-1}}{3\delta^2}\Big[3dy_1 \wedge d\bar{y}_1 + d(y_1 + \delta x_i) \wedge d(\bar{y}_1 + \delta \bar{x}_i)$$
$$-d(2y_1 + \delta x_i/2) \wedge d(2\bar{y}_1 + \delta \bar{x}_i/2)\Big]$$
$$\leq \frac{4\sqrt{-1}}{3\delta^2}\Big[3dy_1 \wedge d\bar{y}_1 + d(y_1 + \delta x_i) \wedge d(\bar{y}_1 + \delta \bar{x}_i)\Big]$$

The last form can be bounded by a combination of $\pi_0^*(\omega)$ and $\pi_{\varepsilon}^*(\omega)$. This completes the proof. □

Proof of Theorem 2.4. We prove that $h_t(f, \mathcal{K}) \leq \log d_t$. We will prove in Paragraphs 2.2 and 2.4 that f admits a totally invariant measure of maximal entropy $\log d_t$ with support in the boundary of $\mathcal{K}$. The variational principle then implies that $h_t(f, \mathcal{K}) = h_t(f, \partial\mathcal{K}) = \log d_t$. We can also conclude using Misiurewicz-Przytycki's theorem 1.109 or Yomdin's theorem 1.110 which can be extended to this case.

Let Γ_n denote the graph of $(f, \ldots, f^{n-1})$ in $V^n \subset (\mathbb{C}^k)^{n-1} \times V$. Let $\pi : (\mathbb{C}^k)^{n-1} \times V \to V$ be the canonical projection. Since $f : U \to V$ is polynomial-like, it is easy to see that $\Gamma_n \subset U^{n-1} \times V$ and that $\pi : \Gamma_n \to V$ defines a ramified covering of degree d_t^n. As in Theorem 1.108, we have

$$h_t(f, \mathcal{K}) \leq \text{lov}(f) := \limsup_{n \to \infty} \frac{1}{n} \log \text{volume}(\Gamma_n \cap \pi^{-1}(U)).$$

But, it follows from Lemma 2.5 that

$$\text{volume}(\Gamma_n \cap \pi^{-1}(U)) \leq c(kn)^s d_t^n.$$

Hence, $\text{lov}(f) \leq \log d_t$. This implies the inequality $h_t(f, \mathcal{K}) \leq \log d_t$. Note that the limit in the definition of $\text{lov}(f)$ exists and we have $\text{lov}(f) = \log d_t$. Indeed, since Γ_n is a covering of degree d_t^n over V, we always have

$$\text{volume}(\Gamma_n \cap \pi^{-1}(U)) \geq d_t^n \text{volume}(U).$$

We show that $d_p \leq d_t$. Let Π_i, $0 \leq i \leq n-1$, denote the projection of V^n onto its factors. We have

$$\text{volume}(\Gamma_n \cap \pi^{-1}(U)) = \sum_{0 \leq i_s \leq n-1} \int_{\Gamma_n \cap \pi^{-1}(U)} \Pi_{i_1}^*(\omega) \wedge \ldots \wedge \Pi_{i_k}^*(\omega).$$

The last sum contains the term

$$\int_{\Gamma_n \cap \pi^{-1}(U)} \Pi_0^*(\omega^{k-p}) \wedge \Pi_{n-1}^*(\omega^p) = \int_{f^{-n+1}(U)} \omega^{k-p} \wedge (f^{n-1})^*(\omega^p).$$

We deduce from the estimate on $\text{volume}(\Gamma_n \cap \pi^{-1}(U))$ and from the definition of d_p that $d_p \leq \text{lov}(f) = d_t$. $\square$

We introduce now others useful dynamical degrees. We call *dynamical $*$-degree of order p* of f the following limit

$$d_p^* := \limsup_{n \to \infty} \sup_S \|(f^n)_*(S)\|_W^{1/n},$$

where $W \Subset V$ is a neighbourhood of $\mathcal{K}$ and the supremum is taken over positive closed $(k-p, k-p)$-current of mass ≤ 1 on a fixed neighbourhood $W' \Subset V$ of $\mathcal{K}$. Clearly, $d_p^* \geq d_p$, since we can take $S = c\omega^{k-p}$ with $c > 0$ small enough.

Lemma 2.6. *The above definition does not depend on the choice of W, W'. Moreover, we have $d_0^* = 1$, $d_k^* = d_t$ and the dynamical $*$-degree of order p of f^n is equal to d_p^{*n}.*

Proof. If N is an integer large enough, the operator $(f^N)_*$ sends continuously positive closed currents on W' to the ones on V. Therefore, the independence of the definition on W' is clear. If S is a probability measure on $\mathcal{K}$, then $(f^n)_*$ is also a probability measure on $\mathcal{K}$. Therefore, $d_0^* = 1$. Observe that

$$\|(f^n)_*(S)\|_W^{1/n} = \left[\int_{f^{-n}(W)} S \wedge (f^n)^*(\omega^p) \right]^{1/n}.$$

So, for the other properties, it is enough to follow the arguments given in Lemma 2.3. $\square$

Many results below are proved under the hypothesis $d_{k-1}^* < d_t$. The following proposition shows that this condition is stable under small perturbations on f. This gives large families of maps satisfying the hypothesis. Indeed, the condition is satisfied for polynomial maps in $\mathbb{C}^k$ which extend at infinity as endomorphisms of $\mathbb{P}^k$. For such maps, if d is the algebraic degree, one can check that $d_p^* \leq d^p$.

Proposition 2.7. *Let $f : U \to V$ be a polynomial-like map of topological degree d_t. Let V' be a convex open set such that $U \Subset V' \Subset V$. If $g : U \to \mathbb{C}^k$ is a holomorphic map, close enough to f and $U' := g^{-1}(V')$, then $g : U' \to V'$ is a polynomial-like map of topological degree d_t. If moreover, f satisfies the condition $d_p^* < d_t$ for some $1 \leq p \leq k-1$, then g satisfies the same property.*

Proof. The first assertion is clear, using the characterization of polynomial-maps by their graphs. We prove the second one. Fix a constant δ with $d_p^* < \delta < d_t$ and an open set W such that $U \Subset W \Subset V$. Fix an integer N large enough such that $\|(f^N)_*(S)\|_W \leq \delta^N$ for any positive closed $(k-p, k-p)$-current S of mass 1 on U. If g is close enough to f, we have $g^{-N}(U) \Subset f^{-N}(W)$ and

$$\|(g^N)^*(\omega^p) - (f^N)^*(\omega^p)\|_{L^\infty(g^{-N}(U))} \leq \varepsilon$$

with $\varepsilon > 0$ a small constant. We have

$$
\begin{aligned}
\|(g^N)_*(S)\|_U &= \int_{g^{-N}(U)} S \wedge (g^N)^*(\omega^p) \\
&\leq \int_{f^{-N}(W)} S \wedge (f^N)^*(\omega^p) + \int_{g^{-N}(U)} S \wedge \left[(g^N)^*(\omega^p) - (f^N)^*(\omega^p)\right] \\
&\leq \|(f^N)_*(S)\|_W + \varepsilon \leq \delta^N + \varepsilon < d_t^N.
\end{aligned}
$$

Therefore, the dynamical $*$-degree $d_p^*(g^N)$ of g^N is strictly smaller than d_t^N. Lemma 2.6 implies that $d_p^*(g) < d_t$. $\qquad\qquad\square$

Remark 2.8. The proof gives that $g \mapsto d_p^*(g)$ is upper semi-continuous on g.

Consider a simple example. Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be the polynomial map $f(z_1, z_2) = (2z_1, z_2^2)$. The restriction of f to $V := \{|z_1| < 2, |z_2| < 2\}$ is polynomial-like and using the current $S = [z_1 = 0]$, it is not difficult to check that $d_1 = d_1^* = d_t = 2$. The example shows that in general one may have $d_{k-1}^* = d_t$.

Exercise 2.9. Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be the polynomial map defined by $f(z_1, z_2) := (3z_2, z_1^2 + z_2)$. Show that the hyperplane at infinity is attracting. Compute the topological degree of f. Compute the topological degree of the map in Example 2.1.

Exercise 2.10. Let f be a polynomial map on $\mathbb{C}^k$ of algebraic degree $d \geq 2$, which extends to a holomorphic endomorphism of $\mathbb{P}^k$. Let V be a ball large enough centered at 0 and $U := f^{-1}(V)$. Prove that the polynomial-like map $f : U \to V$ satisfies $d_p^* = d^p$ and $d_t = d^k$. Hint: use the Green function and Green currents.

2.2 Construction of the Green Measure

In this paragraph, we introduce the first version of the dd^c-method. It allows to construct for a polynomial-like map f a canonical measure which is totally invariant. As we have seen in the case of endomorphisms of $\mathbb{P}^k$, the method gives good estimates and allows to obtain precise stochastic properties. Here, we will see that it applies under a very weak hypothesis. The construction of the measure does not require any hypothesis on the dynamical degrees and give useful convergence results.

Consider a polynomial-like map $f : U \to V$ of topological degree $d_t > 1$ as above. Define the Perron-Frobenius operator Λ acting on test functions φ by

$$\Lambda(\varphi)(z) := d_t^{-1} f_*(\varphi)(z) := d_t^{-1} \sum_{w \in f^{-1}(z)} \varphi(w),$$

where the points in $f^{-1}(z)$ are counted with multiplicity. Since f is a ramified covering, $\Lambda(\varphi)$ is continuous when φ is continuous. If ν is a probability measure on V, define the measure $f^*(\nu)$ by

$$\langle f^*(\nu), \varphi \rangle := \langle \nu, f_*(\varphi) \rangle.$$

This is a positive measure of mass d_t supported on $f^{-1}(\operatorname{supp}(\nu))$. Observe that the operator $\nu \mapsto d_t^{-1} f^*(\nu)$ is continuous on positive measures, see Exercise A.11.

Theorem 2.11. *Let $f : U \to V$ be a polynomial-like map as above. Let ν be a probability measure supported on V which is defined by an L^1 form. Then $d_t^{-n}(f^n)^*(\nu)$ converge to a probability measure   which does not depend on ν. For φ p.s.h. on a neighbourhood of the filled Julia set $\mathscr{K}$, we have $\langle d_t^{-n}(f^n)^*(\nu), \varphi \rangle \to \langle \ \ , \varphi \rangle$. The measure   is supported on the boundary of $\mathscr{K}$ and is totally invariant: $d_t^{-1} f^*(\ ) = f_*(\ ) =\ $. Moreover, if Λ is the Perron-Frobenius operator associated to f and φ is a p.s.h. function on a neighbourhood of $\mathscr{K}$, then $\Lambda^n(\varphi)$ converge to $\langle \ \ , \varphi \rangle$.*

Note that in general $\langle \ \ , \varphi \rangle$ may be $-\infty$. If $\langle \ \ , \varphi \rangle = -\infty$, the above convergence means that $\Lambda^n(\varphi)$ tend locally uniformly to $-\infty$; otherwise, the convergence is in L_{loc}^p for $1 \le p < +\infty$, see Appendix A.2. The above result still holds for measures ν which have no mass on pluripolar sets. The proof in that case is more delicate. We have the following lemma.

Lemma 2.12. *If φ is p.s.h. on a neighbourhood of $\mathscr{K}$, then $\Lambda^n(\varphi)$ converge to a constant c_φ in $\mathbb{R} \cup \{-\infty\}$.*

Proof. Observe that $\Lambda^n(\varphi)$ is defined on V for n large enough. It is not difficult to check that these functions are p.s.h. Indeed, when φ is a continuous p.s.h. function, $\Lambda^n(\varphi)$ is a continuous function, see Exercise A.11, and $dd^c \Lambda^n(\varphi) = d_t^{-n}(f^n)_*(dd^c \varphi) \ge 0$. So, $\Lambda^n(\varphi)$ is p.s.h. The general case is obtained using an approximation of φ by a decreasing sequence of smooth p.s.h. functions.

Consider ψ the upper semi-continuous regularization of $\limsup \Lambda^n(\varphi)$. We deduce from Proposition A.20 that ψ is a p.s.h. function. We first prove that ψ is constant. Assume not. By maximum principle, there is a constant δ such that $\sup_{\overline{U}} \psi < \delta < \sup_V \psi$. By Hartogs' lemma A.20, for n large enough, we have $\Lambda^n(\varphi) < \delta$ on U. Since the fibers of f are contained in U, we deduce from the definition of Λ that

$$\sup_V \Lambda^{n+1}(\varphi) = \sup_V \Lambda(\Lambda^n(\varphi)) \le \sup_U \Lambda^n(\varphi) < \delta.$$

This implies that $\psi \le \delta$ which contradicts the choice of δ. So ψ is constant.

Denote by c_φ this constant. If $c_\varphi = -\infty$, it is clear that $\Lambda^n(\varphi)$ converge to $-\infty$ uniformly on compact sets. Assume that c_φ is finite and $\Lambda^{n_i}(\varphi)$ does not converge to c_φ for some sequence (n_i). By Hartogs' lemma, we have $\Lambda^{n_i}(\varphi) \le c_\varphi - \varepsilon$ for some constant $\varepsilon > 0$ and for i large enough. We deduce as above that $\Lambda^n(\varphi) \le c_\varphi - \varepsilon$ for $n \ge n_i$. This contradicts the definition of c_φ. $\qquad\square$

Proof of Theorem 2.11. We can replace v with $d_t^{-1} f^*(v)$ in order to assume that v is supported on U. The measure v can be written as a finite or countable sum of bounded positive forms, we can assume that v is a bounded form.

Consider a smooth p.s.h. function φ on a neighbourhood of $\mathcal{K}$. It is clear that $\Lambda^n(\varphi)$ are uniformly bounded for n large enough. Therefore, the constant c_φ is finite. We deduce from Lemma 2.12 that $\Lambda^n(\varphi)$ converge in $L^1_{loc}(V)$ to c_φ. It follows that

$$\langle d_t^{-n}(f^n)^*(v), \varphi \rangle = \langle v, \Lambda^n(\varphi) \rangle \to c_\varphi.$$

Let ϕ be a general smooth function on V. We can always write ϕ as a difference of p.s.h. functions on U. Therefore, $\langle d_t^{-n}(f^n)^*(v), \phi \rangle$ converge. It follows that the sequence of probability measures $d_t^{-n}(f^n)^*(v)$ converges to some probability measure   . Since c_φ does not depend on v, the measure   does not depend on v. Consider a measure v supported on $U \setminus \mathcal{K}$. So, the limit   of $d_t^{-n}(f^n)^*(v)$ is supported on $\partial\mathcal{K}$. The total invariance is a direct consequence of the above convergence.

For the rest of the theorem, assume that φ is a general p.s.h. function on a neighbourhood of $\mathcal{K}$. Since $\limsup \Lambda^n(\varphi) \leq c_\varphi$, Fatou's lemma implies that

$$\langle \ , \varphi \rangle = \langle d_t^{-n}(f^n)^*(\ ), \varphi \rangle = \langle \ , \Lambda^n(\varphi) \rangle \leq \langle \ , \limsup_{n \to \infty} \Lambda^n(\varphi) \rangle = c_\varphi.$$

On the other hand, for v smooth on U, we have since φ is upper semi-continuous

$$c_\varphi = \lim_{n \to \infty} \langle v, \Lambda^n(\varphi) \rangle = \lim_{n \to \infty} \langle d_t^{-n}(f^n)^*(v), \varphi \rangle \leq \langle \lim_{n \to \infty} d_t^{-n}(f^n)^*(v), \varphi \rangle = \langle \ , \varphi \rangle.$$

Therefore, $c_\varphi = \langle \ , \varphi \rangle$. Hence, $\Lambda^n(\varphi)$ converge to $\langle \ , \varphi \rangle$ for an arbitrary p.s.h. function φ. This also implies that $\langle d_t^{-n}(f^n)^*(v), \varphi \rangle \to \langle \ , \varphi \rangle$.   □

The measure   is called *the equilibrium measure* of f. We deduce from the above arguments the following result.

Proposition 2.13. *Let v be a totally invariant probability measure. Then v is supported on $\mathcal{K}$. Moreover, $\langle v, \varphi \rangle \leq \langle \ , \varphi \rangle$ for every function φ which is p.s.h. in a neighbourhood of $\mathcal{K}$ and $\langle v, \varphi \rangle = \langle \ , \varphi \rangle$ if φ is pluriharmonic in a neighbourhood of $\mathcal{K}$.*

Proof. Since $v = d_t^{-n}(f^n)^*(v)$, it is supported on $f^{-n}(V)$ for every $n \geq 0$. So, v is supported on $\mathcal{K}$. We know that $\limsup \Lambda^n(\varphi) \leq c_\varphi$, then Fatou's lemma implies that

$$\langle v, \varphi \rangle = \lim_{n \to \infty} \langle d_t^{-n}(f^n)^*(v), \varphi \rangle = \lim_{n \to \infty} \langle v, \Lambda^n(\varphi) \rangle \leq c_\varphi.$$

When φ is pluriharmonic, the inequality holds for $-\varphi$; we then deduce that $\langle v, \varphi \rangle \geq c_\varphi$. The proposition follows.   □

Corollary 2.14. *Let X_1, X_2 be two analytic subsets of V such that $f^{-1}(X_1) \subset X_1$ and $f^{-1}(X_2) \subset X_2$. Then $X_1 \cap X_2 \neq \varnothing$. In particular, f admits at most one point a such that $f^{-1}(a) = \{a\}$.*

Proof. Observe that there are totally invariant probability measures v_1, v_2 supported on X_1, X_2. Indeed, if v is a probability measure supported on X_i, then any limit value of

$$\frac{1}{N} \sum_{n=0}^{N-1} d_t^{-n} (f^n)^*(v)$$

is supported on X_i and is totally invariant. We are using the continuity of the operator $v \mapsto d_t^{-1} f^*(v)$, for the weak topology on measures.

On the other hand, if X_1 and X_2 are disjoint, we can find a holomorphic function h on U such that $h = c_1$ on X_1 and $h = c_2$ on X_2, where c_1, c_2 are distinct constants. We consider the function defined on $X_1 \cup X_2$ as claimed and extend it as a holomorphic function in U. This is possible since V is convex [HO2]. Adding to h a constant allows to assume that h does not vanish on $\mathscr{K}$. Therefore, $\varphi := \log |h|$ is pluriharmonic on a neighbourhood of $\mathscr{K}$. We have $\langle v_1, \varphi \rangle \neq \langle v_2, \varphi \rangle$. This contradicts Proposition 2.13. $\qquad\square$

When the test function is pluriharmonic, we have the following exponential convergence.

Proposition 2.15. *Let W be a neighbourhood of $\mathscr{K}$ and $\mathscr{F}$ a bounded family of pluriharmonic functions on W. There are constants $N \geq 0$, $c > 0$ and $0 < \lambda < 1$ such that if φ is a function in $\mathscr{F}$, then*

$$|\langle \Lambda^n(\varphi) - \langle \ , \varphi \rangle| \leq c\lambda^n \quad on \quad V$$

for $n \geq N$.

Proof. Observe that if N is large enough, the functions $\Lambda^N(\varphi)$ are pluriharmonic and they absolute values are bounded on V by the same constant. We can replace φ with $\Lambda^N(\varphi)$ in order to assume that $W = V$, $N = 0$ and that $|\varphi|$ is bounded by some constant A. Then, $|\Lambda^n(\varphi)| \leq A$ for every n. Subtracting from φ the constant $\langle \ , \varphi \rangle$ allows to assume that $\langle \ , \varphi \rangle = 0$.

Let $\mathscr{F}_\alpha$ denote the family of pluriharmonic functions φ on V such that $|\varphi| \leq \alpha$ and $\langle \ , \varphi \rangle = 0$. It is enough to show that Λ sends $\mathscr{F}_\alpha$ into $\mathscr{F}_{\lambda\alpha}$ for some constant $0 < \lambda < 1$. We can assume $\alpha = 1$. Since   is totally invariant, Λ preserves the subspace $\{\varphi, \langle \ , \varphi \rangle = 0\}$. The family $\mathscr{F}_1$ is compact and does not contain the function identically equal to 1. By maximum principle applied to $\pm\varphi$, there is a constant $0 < \lambda < 1$ such that $\sup_U |\varphi| \leq \lambda$ for φ in $\mathscr{F}_1$. We deduce that $\sup_V |\Lambda(\varphi)| \leq \lambda$. The result follows. $\qquad\square$

The following result shows that the equilibrium measure   satisfies the Oseledec's theorem hypothesis. It can be extended to a class of orientation preserving smooth maps on Riemannian manifolds [DS1].

Theorem 2.16. *Let $f : U \to V$ be a polynomial-like map as above. Let   be the equilibrium measure and J the Jacobian of f with respect to the standard volume form on $\mathbb{C}^k$. Then*

$$\langle \ , \log J \rangle \geq \log d_t.$$

In particular,   has no mass on the critical set of f.

Proof. Let ν be the restriction of the Lebesgue measure to U multiplied by a constant so that $\|\nu\| = 1$. Define

$$\nu_n := d_t^{-n}(f^n)^*(\nu) \quad \text{and} \quad \mu_N := \frac{1}{N}\sum_{n=1}^{N}\nu_n.$$

By Theorem 2.11, μ_N converge to μ. Choose a constant $M > 0$ such that $J \leq M$ on U. For any constant $m > 0$, define

$$g_m(x) := \min\left(\log\frac{M}{J(x)}, m + \log M\right) = \min\left(\log\frac{M}{J(x)}, m'\right)$$

with $m' := m + \log M$. This is a family of continuous functions which are positive, bounded on U and which converge to $\log M/J$ when m goes to infinity. Define

$$s_N(x) := \frac{1}{N}\sum_{q=0}^{N-1}g_m(f^q(x)).$$

Using the definition of f^* on measures, we obtain

$$\langle \nu_N, s_N\rangle = \frac{1}{N}\sum_{q=0}^{N-1}d_t^{-N}\langle(f^N)^*(\nu), g_m\circ f^q\rangle$$

$$= \frac{1}{N}\sum_{q=0}^{N-1}d_t^{-N+q}\langle(f^{N-q})^*(\nu), g_m\rangle$$

$$= \frac{1}{N}\sum_{q=0}^{N-1}\langle\nu_{N-q}, g_m\rangle = \langle\mu_N, g_m\rangle.$$

In order to bound $\langle\mu, \log J\rangle$ from below, we will bound $\langle\mu_N, g_m\rangle$ from above.

For $\alpha > 0$, let U_N^α denote the set of points $x \in U$ such that $s_N(x) > \alpha$. Since $s_N(x) \leq m'$, we have

$$\langle\mu_N, g_m\rangle = \langle\nu_N, s_N\rangle \leq m'\nu_N(U_N^\alpha) + \alpha(1 - \nu_N(U_N^\alpha))$$
$$= \alpha + (m' - \alpha)\nu_N(U_N^\alpha).$$

If $\nu_N(U_N^\alpha)$ converge to 0 when $N \to \infty$, then

$$\langle\mu, g_m\rangle = \lim_{N\to\infty}\langle\mu_N, g_m\rangle \leq \alpha \quad \text{and hence} \quad \langle\mu, \log M/J\rangle \leq \alpha.$$

We determine a value of α such that $\nu_N(U_N^\alpha)$ tend to 0.

By definition of ν_N, we have

$$\nu_N(U_N^\alpha) = \int_{U_N^\alpha}d_t^{-N}(f^N)^*(\nu) = \int_{U_N^\alpha}d_t^{-N}\left(\prod_{q=0}^{N-1}J\circ f^q\right)d\nu.$$

Define for a given $\delta > 0$ and for any integer j,

$$W_j := \big\{ \exp(-j\delta) < J \le \exp(-(j-1)\delta) \big\}$$

and

$$\tau_j(x) := \frac{1}{N} \#\big\{ q, \quad f^q(x) \in W_j \text{ and } 0 \le q \le N-1 \big\}.$$

We have $\sum \tau_j = 1$ and

$$\nu_N(U_N^\alpha) \le \int_{U_N^\alpha} \left[\frac{1}{d_t} \exp\left(\sum -(j-1)\delta\tau_j \right) \right]^N dv.$$

Using the inequality $g_m \le \log M/J$, we have on U_N^α

$$\alpha < s_N < \sum \tau_j (\log M + j\delta) = \sum j\delta\tau_j + \log M.$$

Therefore,

$$-\sum (j-1)\delta\tau_j < -\alpha + (\log M + \delta).$$

We deduce from the above estimate on $\nu_N(U_N^\alpha)$ that

$$\nu_N(U_N^\alpha) \le \int_{U_N^\alpha} \left[\frac{\exp(-\alpha)M\exp(\delta)}{d_t} \right]^N dv.$$

So, for every $\alpha > \log(M/d_t) + \delta$, we have $\nu_N(U_N^\alpha) \to 0$.

Choosing δ arbitrarily small, we deduce from the above discussion that

$$\lim_{N\to\infty} \langle \ _N, g_m \rangle \le \log(M/d_t).$$

Since g_m is continuous and $_N$ converge to , we have $\langle \ , g_m \rangle \le \log(M/d_t)$. Letting m go to infinity gives $\langle \ , \log J \rangle \ge \log d_t$. $\qquad\qquad \square$

Exercise 2.17. Let φ be a strictly p.s.h. function on a neighbourhood of $\mathscr{K}$, i.e. a p.s.h. function satisfying $dd^c\varphi \ge cdd^c\|z\|^2$, with $c > 0$, in a neighbourhood of $\mathscr{K}$. Let ν be a probability measure such that $\langle d_t^{-n}(f^n)^*(\nu), \varphi \rangle$ converge to $\langle \ , \varphi \rangle$ and that $\langle \ , \varphi \rangle$ is finite. Show that $d_t^{-n}(f^n)^*(\nu)$ converge to .

Exercise 2.18. Using the test function $\varphi = \|z\|^2$, show that

$$\int_{f^{-n}(U)} (f^n)^*(\omega^{k-1}) \wedge \omega = o(d_t^n),$$

when n goes to infinity.

Exercise 2.19. Let Y denote the set of critical values of f. Show that the volume of $f^n(Y)$ in U satisfies $\mathrm{volume}(f^n(Y) \cap U) = o(d_t^n)$ when n goes to infinity.

Exercise 2.20. Let f be a polynomial endomorphism of $\mathbb{C}^2$ of algebraic degree $d \geq 2$. Assume that f extends at infinity to an endomorphism of $\mathbb{P}^2$. Show that f admits at most three totally invariant points[7].

2.3 Equidistribution Problems

In this paragraph, we consider polynomial-like maps f satisfying the hypothesis that the dynamical degree d_{k-1}^* is strictly smaller than d_t. We say that f has a *large topological degree*. We have seen that this property is stable under small pertubations of f. Let Y denote the hypersurface of critical values of f. As in the case of endomorphisms of $\mathbb{P}^k$, define the ramification current R by

$$R := \sum_{n \geq 0} d_t^{-n} (f^n)_* [Y].$$

The following result is a version of Proposition 1.51.

Proposition 2.21. *Let $f : U \to V$ be a polynomial-like map as above with large topological degree. Let ν be a strictly positive constant and let a be a point in V such that the Lelong number $\nu(R, a)$ is strictly smaller than ν. Let δ be a constant such that $d_{k-1} < \delta < d_t$. Then, there is a ball B centered at a such that f^n admits at least $(1 - \sqrt{\nu})d_t^n$ inverse branches $g_i : B \to W_i$ where W_i are open sets in V of diameter $\leq \delta^{n/2} d_t^{-n/2}$. In particular, if $\ '$ is a limit value of the measures $d_t^{-n}(f^n)^*(\delta_a)$ then $\| \ ' - \ \| \leq 2\sqrt{\nu(R, a)}$.*

Proof. Since $d_{k-1}^* < d_t$, the current R is well-defined and has locally finite mass. If ω is the standard Kähler form on $\mathbb{C}^k$, we also have $\|(f^n)_*(\omega)\|_{V'} \lesssim \delta^n$ for every open set $V' \Subset V$. So, for the first part of the proposition, it is enough to follow the arguments in Proposition 1.51. The proof there is written in such way that the estimates are local and can be extended without difficulty to the present situation. In particular, we did not use Bézout's theorem.

For the second assertion, we do not have yet the analogue of Proposition 1.46, but it is enough to compare $d_t^{-n}(f^n)^*(\delta_a)$ with the pull-backs of a smooth measure supported on B and to apply Theorem 2.11. □

We deduce the following result as in the case of endomorphisms of $\mathbb{P}^k$.

Theorem 2.22. *Let $f : U \to V$ be a polynomial-like map as above with large topological degree. Let P_n denote the set of repelling periodic points of period n on the support of . Then the sequence of measures*

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[7] This result was proved in [DS1]. Amerik and Campana proved in [AC] for a general non-invertible endomorphism of $\mathbb{P}^2$ that the number of totally invariant points is at most equal to 9. The sharp bound (probably 3) is unknown. One deduces from Corollary 2.14 and Amerik-Campana result that any non-invertible polynomial map on $\mathbb{C}^3$ which extends at infinity to an endomorphism of $\mathbb{P}^3$, admits at most 10 totally invariant points.

$$n := d_t^{-n} \sum_{a \in P_n} \delta_a$$

converges to   .

Proof. It is enough to repeat the proof of Theorem 1.57 and to show that f^n admits exactly d_t^n fixed points counted with multiplicity. We can assume $n = 1$. Let Γ denote the graph of f in $U \times V \subset V \times V$. The number of fixed points of f is the number of points in the intersection of Γ with the diagonal Δ of $\mathbb{C}^k \times \mathbb{C}^k$. Observe that this intersection is contained in the compact set $\mathscr{K} \times \mathscr{K}$. For simplicity, assume that V contains the point 0 in $\mathbb{C}^k$. Let (z, w) denote the standard coordinates on $\mathbb{C}^k \times \mathbb{C}^k$ where the diagonal is given by the equation $z = w$. Consider the deformations $\Delta_t := \{w = tz\}$ with $0 \leq t \leq 1$ of Δ. Since V is convex, it is not difficult to see that the intersection of Γ with this family stays in a compact subset of $V \times V$. Therefore, the number of points in $\Delta_t \cap \Gamma$, counted with multiplicity, does not depend on t. For $t = 0$, this is just the number of points in the fiber $f^{-1}(0)$. The result follows. $\square$

The equidistribution of negative orbits of points is more delicate than in the case of endomorphisms of $\mathbb{P}^k$. It turns out that the exceptional set $\mathscr{E}$ does not satisfy in general $f^{-1}(\mathscr{E}) = \mathscr{E} \cap U$. We have the following result.

Theorem 2.23. *Let $f : U \to V$ be a polynomial-like map as above with large topological degree. Then there is a proper analytic subset $\mathscr{E}$ of V, possibly empty, such that $d_t^{-n}(f^n)^*(\delta_a)$ converge to the equilibrium measure   if and only if a does not belong to the orbit of $\mathscr{E}$. Moreover, $\mathscr{E}$ satisfies $f^{-1}(\mathscr{E}) \subset \mathscr{E} \subset f(\mathscr{E})$ and is maximal in the sense that if E is a proper analytic subset of V contained in the orbit of critical values such that $f^{-n}(E) \subset E$ for some $n \geq 1$ then $E \subset \mathscr{E}$.*

The proof follows the main lines of the case of endomorphisms of $\mathbb{P}^k$ using the following proposition applied to Z the set of critical values of f. The set $\mathscr{E}$ will be defined as $\mathscr{E} := \mathscr{E}_Z$. Observe that unlike in the case of endomorphisms of $\mathbb{P}^k$, we need to assume that E is in the orbit of the critical values.

Let Z be an arbitrary analytic subset of V not necessarily of pure dimension. Let $N_n(a)$ denote the number of orbits

$$a_{-n}, \ldots, a_{-1}, a_0$$

with $f(a_{-i-1}) = a_{-i}$ and $a_0 = a$ such that $a_{-i} \in Z$ for every i. Here, the orbits are counted with multiplicity, i.e. we count the multiplicity of f^n at a_{-n}. So, $N_n(a)$ is the number of negative orbits of order n of a which stay in Z. Observe that the sequence of functions $\tau_n := d_t^{-n} N_n$ decreases to some function τ. Since τ_n are upper semi-continuous with respect to the Zariski topology and $0 \leq \tau_n \leq 1$, the function τ satisfies the same properties. Note that $\tau(a)$ is the proportion of infinite negative orbits of a staying in Z. Define $\mathscr{E}_Z := \{\tau = 1\}$. The Zariski upper semi-continuity of τ implies that $\mathscr{E}_Z$ is analytic. It is clear that $f^{-1}(\mathscr{E}_Z) \subset \mathscr{E}_Z$ which implies that $\mathscr{E}_Z \subset f(\mathscr{E}_Z)$.

Proposition 2.24. *If a point $a \in V$ does not belong to the orbit $\cup_{n \geq 0} f^n(\mathscr{E}_Z)$ of $\mathscr{E}_Z$, then $\tau(a) = 0$.*

Proof. Assume there is $\theta_0 > 0$ such that $\{\tau \geq \theta_0\}$ is not contained in the orbit of $\mathscr{E}_Z$. We claim that there is a maximal value θ_0 satisfying the above property. Indeed, by definition, $\tau(a)$ is smaller than or equal to the average of τ on the fiber of a. So, we only have to consider the components of $\{\tau \geq \theta_0\}$ which intersect U and there are only finitely many of such components, hence the maximal value exists.

Let E be the union of irreducible components of $\{\tau \geq \theta_0\}$ which are not contained in the orbit of $\mathscr{E}_Z$. Since $\theta_0 > 0$, we know that $E \subset Z$. We want to prove that E is empty. If a is a generic point in E, it does not belong to the orbit of $\mathscr{E}_Z$ and we have $\tau(a) = \theta_0$. If b is a point in $f^{-1}(a)$, then b is not in the orbit of $\mathscr{E}_Z$. Therefore, $\tau(b) \leq \theta_0$. Since $\tau(a)$ is smaller than or equal to the average of τ on $f^{-1}(a)$, we deduce that $\tau(b) = \theta_0$, and hence $f^{-1}(a) \subset E$. By induction, we obtain that $f^{-n}(a) \subset E \subset Z$ for every $n \geq 1$. Hence, $a \in \mathscr{E}_Z$. This is a contradiction.   □

The following example shows that in general the orbit of $\mathscr{E}$ is not an analytic set. We deduce that in general polynomial-like maps are not homeomorphically conjugated to restrictions on open sets of endomorphisms of $\mathbb{P}^k$ (or polynomial maps of $\mathbb{C}^k$ such that the infinity is attractive) with the same topological degree.

Example 2.25. Denote by $D(a, R)$ the disc of radius R and of center a in $\mathbb{C}$. Observe that the polynomial $P(z) := 6z^2 + 1$ defines a ramified covering of degree 2 from $D := P^{-1}(D(0,4))$ to $D(0,4)$. The domain D is simply connected and is contained in $D(0,1)$. Let ψ be a bi-holomorphic map between $D(1,2)$ and $D(0,1)$ such that $\psi(0) = 0$. Define $h(z,w) := (P(z), 4\psi^m(w))$ with m large enough. This application is holomorphic and proper from $W := D \times D(1,2)$ to $V := D(0,4) \times D(0,4)$. Its critical set is given by $zw = 0$.

Define also

$$g(z,w) := 10^{-2}\big(\exp(z)\cos(\pi w/2), \exp(z)\sin(\pi w/2)\big).$$

One easily check that g defines a bi-holomorphic map between W and $U := g(W)$. Consider now the polynomial-like map $f : U \to V$ defined by $f = h \circ g^{-1}$. Its topological degree is equal to $2m$; its critical set C is equal to $g\{zw = 0\}$. The image of $C_1 := g\{z = 0\}$ by f is equal to $\{z = 1\}$ which is outside U. The image of $g\{w = 0\}$ by f is $\{w = 0\} \cap V$.

The intersection $\{w = 0\} \cap U$ contains two components $C_2 := g\{w = 0\}$ and $C_2' := g\{w = 2\}$. They are disjoint because g is bi-holomorphic. We also have $f(C_2) = \{w = 0\} \cap V$ and $f^{-1}\{w = 0\} = C_2$. Therefore, $\mathscr{E} = \{w = 0\} \cap V$, $f^{-1}(\mathscr{E}) \subset \mathscr{E}$ and $f^{-1}(\mathscr{E}) \neq \mathscr{E} \cap U$ since $f^{-1}(\mathscr{E})$ does not contain C_2'. The orbit of $\mathscr{E}$ is the union of C_2 and of the orbit of C_2'. Since m is large, the image of C_2' by f is a horizontal curve very close to $\{w = 0\}$. It follows that the orbit of C_2' is a countable union of horizontal curves close to $\{w = 0\}$ and it is not analytic.

It follows that f is not holomorphically conjugate to an endomorphism (or a polynomial map such that the hyperplane at infinity is attractive) with the same topological degree. If it were, the exceptional set would not have infinitely many components in a neighbourhood of $w = 0$.

Remark 2.26. Assume that f is not with large topological degree but that the series which defines the ramification current R converges. We can then construct inverse branches as in Proposition 2.21. To obtain the same exponential estimates on the diameter of W_i, it is enough to assume that $d_{k-1} < d_t$. In general, we only have that these diameters tend uniformly to 0 when n goes to infinity. Indeed, we can use the estimate in Exercise 2.18. The equidistribution of periodic points and of negative orbits still holds in this case.

Exercise 2.27. Let f be a polynomial-like map with large topological degree. Show that there is a small perturbation of f, arbitrarily close to f, whose exceptional set is empty.

2.4 Properties of the Green Measure

Several properties of the equilibrium measure of polynomial-like maps can be proved using the arguments that we introduced in the case of endomorphisms of $\mathbb{P}^k$. We have the following result for general polynomial-like maps.

Theorem 2.28. *Let* $f : U \to V$ *be a polynomial-like map of topological degree* $d_t > 1$. *Then its equilibrium measure is an invariant measure of maximal entropy* $\log d_t$. *Moreover, is K-mixing.*

Proof. By Theorem 2.16, has no mass on the critical set of f. Therefore, it is an invariant measure of constant Jacobian d_t in the sense that $(f(A)) = d_t$ (A) when f is injective on a Borel set A. We deduce from Parry's theorem 1.116 that $h(f) \geq \log d_t$. The variational principle and Theorem 2.4 imply that $h(f) = \log d_t$.

We prove the K-mixing property. As in the case of endomorphisms of $\mathbb{P}^k$, it is enough to show for φ in $L^2()$ that $\Lambda^n(\varphi) \to \langle \ , \varphi \rangle$ in $L^2()$. Since $\Lambda : L^2() \to L^2()$ is of norm 1, it is enough to check the convergence for a dense family of φ. So, we only have to consider φ smooth. We can also assume that φ is p.s.h. because smooth functions can be written as a difference of p.s.h. functions. Assume also for simplicity that $\langle \ , \varphi \rangle = 0$.

So, the p.s.h. functions $\Lambda^n(\varphi)$ converge to 0 in $L^p_{loc}(V)$. By Hartogs' lemma A.20, $\sup_U \Lambda^n(\varphi)$ converge to 0. This and the identity $\langle \ , \Lambda^n(\varphi) \rangle = \langle \ , \varphi \rangle = 0$ imply that $\{\Lambda^n(\varphi) < -\delta\} \to 0$ for every fixed $\delta > 0$. On the other hand, by definition of Λ, $|\Lambda^n(\varphi)|$ is bounded by $\|\varphi\|_\infty$ which is a constant independent of n. Therefore, $\Lambda^n(\varphi) \to 0$ in $L^2()$ and K-mixing follows. □

The following result, due to Saleur [S], generalizes Theorem 1.118.

Theorem 2.29. *Let* $f : U \to V$ *be a polynomial-like map with large topological degree. Then its equilibrium measure is the unique invariant probability measure of maximal entropy.*

The proof follows the one of Theorem 1.118. The estimates of some integrals over $\mathbb{P}^k$ are replaced with the volume estimate in Lemma 2.5. A new difficulty here is to prove that if v is an ergodic measure of maximal entropy, it has no mass on the set of critical values of f which may be singular. For this purpose, Saleur uses a version of Lemma 2.5 where V is replaced with an analytic set.

Theorem 2.30. *Let f and   be as above. Then the sum of the Lyapounov exponents of   is at least equal to $\frac{1}{2}\log d_t$. In particular, f admits a strictly positive Lyapounov exponent. If f is with large topological degree, then   is hyperbolic and its Lyapounov exponents are at least equal to $\frac{1}{2}\log(d_t/d_{k-1})$.*

Proof. By Oseledec's theorem 1.119, applied in the complex setting, the sum of the Lyapounov exponents of   (associated to complex linear spaces) is equal to $\frac{1}{2}\langle$ $,\log J\rangle$. Theorem 2.16 implies that this sum is at least equal to $\frac{1}{2}\log d_t$. The second assertion is proved as in Theorem 1.120 using Proposition 2.21.   □

From now on, we only consider maps with large topological degree. The following result was obtained by Dinh-Dupont in [DD]. It generalizes Theorem 1.121.

Theorem 2.31. *Let f be a polynomial-like map with large topological degree as above. Let $\chi_1,\ldots,\chi_k$ denote the Lyapounov exponents of the equilibrium measure   ordered by $\chi_1 \geq \cdots \geq \chi_k$ and Σ their sum. Then the Hausdorff dimension of   satisfies*

$$\frac{\log d_t}{\chi_1} \leq \dim_H(\ ) \leq 2k - \frac{2\Sigma - \log d_t}{\chi_1}.$$

We now prove some stochastic properties of the equilibrium measure. We first introduce some notions. Let V be an open subset of $\mathbb{C}^k$ and v a probability measure with compact support in V. We consider v as a function on the convex cone $\mathrm{PSH}(V)$ of p.s.h. functions on V, with the L^1_{loc}-topology. We say that v is *PB* if this function is finite, i.e. p.s.h. functions on V are v-integrable. We say that v is *PC* if it is PB and defines a continuous functional on $\mathrm{PSH}(V)$. Recall that the weak topology on $\mathrm{PSH}(V)$ coincides with the L^p_{loc} topology for $1 \leq p < +\infty$. In dimension 1, a measure is PB if it has locally bounded potentials, a measure is PC if it has locally continuous potentials. A measure v is *moderate* if for any bounded subset $\mathscr{P}$ of $\mathrm{PSH}(V)$, there are constants $\alpha > 0$ and $A > 0$ such that

$$\langle v, e^{\alpha|\varphi|}\rangle \leq A \quad \text{for} \quad \varphi \in \mathscr{P}.$$

Let K be a compact subset of V. Define a *pseudo-distance* $\mathrm{dist}_{L^1(K)}$ between φ, ψ in $\mathrm{PSH}(V)$ by

$$\mathrm{dist}_{L^1(K)}(\varphi, \psi) := \|\varphi - \psi\|_{L^1(K)}.$$

Observe that if v is continuous with respect to $\mathrm{dist}_{L^1(K)}$ then v is PC. The following proposition gives a criterion for a measure to be moderate.

Proposition 2.32. *If v is Hölder continuous with respect to $\mathrm{dist}_{L^1(K)}$ for some compact subset K of V, then v is moderate. If v is moderate, then p.s.h. functions on V are in $L^p(v)$ for every $1 \leq p < +\infty$ and v has positive Hausdorff dimension.*

Proof. We prove the first assertion. Assume that ν is Hölder continuous with respect to $\mathrm{dist}_{L^1(K)}$ for some compact subset K of V. Consider a bounded subset $\mathscr{P}$ of $\mathrm{PSH}(V)$. The functions in $\mathscr{P}$ are uniformly bounded above on K. Therefore, subtracting from these functions a constant allows to assume that they are negative on K. We want to prove the estimate $\langle \nu, e^{-\alpha\varphi}\rangle \leq A$ for $\varphi \in \mathscr{P}$ and for some constants α, A. It is enough to show that $\nu\{\varphi < -M\} \lesssim e^{-\alpha M}$ for some (other) constant $\alpha > 0$ and for $M \geq 1$. For $M \geq 0$ and $\varphi \in \mathscr{P}$, define $\varphi_M := \max(\varphi, -M)$. We replace $\mathscr{P}$ with the family of functions φ_M. This allows to assume that the family is stable under the operation $\max(\cdot, -M)$. Observe that $\varphi_{M-1} - \varphi_M$ is positive, supported on $\{\varphi < -M+1\}$, smaller or equal to 1, and equal to 1 on $\{\varphi < -M\}$. In order to obtain the above estimate, we only have to show that $\langle \nu, \varphi_{M-1} - \varphi_M\rangle \lesssim e^{-\alpha M}$ for some $\alpha > 0$.

Fix a constant $\lambda > 0$ small enough and a constant $A > 0$ large enough. Since ν is Hölder continuous and $\varphi_{M-1} - \varphi_M$ vanishes on $\{\varphi > -M+1\}$, we have

$$\nu\{\varphi < -M\} \leq \langle \nu, \varphi_{M-1}\rangle - \langle \nu, \varphi_M\rangle \leq A \left\|\varphi_{M-1} - \varphi_M\right\|_{L^1(K)}^{\lambda}$$
$$\leq A\,\mathrm{volume}\{\varphi \leq -M+1\}^{\lambda}.$$

On the other hand, since $\mathscr{P}$ is a bounded family in $\mathrm{PSH}(V)$, by Theorem A.22, we have $\|e^{-\lambda\varphi}\|_{L^1(K)} \leq A$ for $\varphi \in \mathscr{P}$. Hence, we have on K

$$\mathrm{volume}\{\varphi \leq -M+1\} \leq Ae^{-\lambda(M-1)}.$$

This implies the desired estimate for $\alpha = \lambda^2$ and completes the proof of the first assertion.

Assume now that ν is moderate. Let φ be a p.s.h. function on V. Then $e^{\alpha|\varphi|}$ is in $L^1(\nu)$ for some constant $\alpha > 0$. Since $e^{\alpha x} \gtrsim x^p$ for $1 \leq p < +\infty$, we deduce that φ is in $L^p(\nu)$. For the last assertion in the proposition, it is enough to show that $\nu(B_r) \leq Ar^{\alpha}$ for any ball B_r of radius $r > 0$ where A, α are some positive constants. We can assume that B_r is a small ball centered at a point $a \in K$. Define $\varphi(z) := \log\|z - a\|$. This function belongs to a compact family of p.s.h. functions. Therefore, $\|e^{-\alpha\varphi}\|_{L^1(\nu)} \leq A$ for some positive constants A, α independent of B_r. Since $e^{-\alpha\varphi} \geq r^{-\alpha}$ on B_r, we deduce that $\nu(B_r) \leq Ar^{\alpha}$. It is well-known that in order to compute the Hausdorff dimension of a set, it is enough to use only coverings by balls. It follows easily that if a Borel set has positive measure, then its Hausdorff dimension is at least equal to α. This completes the proof. $\qquad\square$

The following results show that the equilibrium measure of a polynomial-like map with large topological degree satisfies the above regularity properties.

Theorem 2.33. *Let $f : U \to V$ be a polynomial-like map. Then the following properties are equivalent:*

1. *The map f has large topological degree, i.e. $d_t > d_{k-1}^*$;*
2. *The measure is PB, i.e. p.s.h. functions on V are integrable with respect to ;*
3. *The measure is PC, i.e. can be extended to a linear continuous form on the cone of p.s.h. functions on V;*

Moreover, if f is such a map, then there is a constant $0 < \lambda < 1$ such that

$$\sup_V \Lambda(\varphi) - \langle \ , \varphi \rangle \leq \lambda \left[\sup_V \varphi - \langle \ , \varphi \rangle \right],$$

for φ p.s.h. on V.

Proof. It is clear that 3) $\Rightarrow$ 2). We show that 1) $\Rightarrow$ 3) and 2) $\Rightarrow$ 1).

1) $\Rightarrow$ 3). Let φ be a p.s.h. function on V. Let W be a convex open set such that $U \Subset W \Subset V$. For simplicity, assume that $\|\varphi\|_{L^1(W)} \leq 1$. So, φ belongs to a compact subset of PSH(W). Therefore, $S := dd^c \varphi$ has locally bounded mass in W. Define $S_n := (f^n)_*(S)$. Fix a constant $\delta > 1$ such that $d_{k-1}^* < \delta < d_t$. Condition 1) implies that $\|(f^n)_*(S)\|_W \lesssim \delta^n$. By Proposition A.16, there are p.s.h. functions φ_n on U such that $dd^c \varphi_n = S_n$ and $\|\varphi_n\|_U \lesssim \delta^n$ on U.

Define $\psi_0 := \varphi - \varphi_0$ and $\psi_n := f_*(\varphi_{n-1}) - \varphi_n$. Observe that these functions are pluriharmonic on U and depend continuously on φ. Moreover, f_* sends continuously p.s.h. functions on U to p.s.h. functions on V. Hence,

$$\|\psi_n\|_{L^1(U)} \leq \|f_*(\varphi_{n-1})\|_{L^1(U)} + \|\varphi_n\|_{L^1(U)} \lesssim \delta^n.$$

We have

$$\begin{aligned}
\Lambda^n(\varphi) &= \Lambda^n(\psi_0 + \varphi_0) = \Lambda^n(\psi_0) + d_t^{-1}\Lambda^{n-1}(f_*(\varphi_0)) \\
&= \Lambda^n(\psi_0) + d_t^{-1}\Lambda^{n-1}(\psi_1 + \varphi_1) = \cdots \\
&= \Lambda^n(\psi_0) + d_t^{-1}\Lambda^{n-1}(\psi_1) + \cdots + d_t^{-n+1}\Lambda(\psi_{n-1}) + d_t^{-n}\psi_n + d_t^{-n}\varphi_n.
\end{aligned}$$

The last term in the above sum converges to 0. The above estimate on ψ_n and their pluriharmonicity imply, by Proposition 2.15, that the sum

$$\Lambda^n(\psi_0) + d_t^{-1}\Lambda^{n-1}(\psi_1) + \cdots + d_t^{-n+1}\Lambda(\psi_{n-1}) + d_t^{-n}\psi_n$$

converges uniformly to the finite constant

$$\langle \ , \psi_0 \rangle + d_t^{-1}\langle \ , \psi_1 \rangle + \cdots + d_t^{-n}\langle \ , \psi_n \rangle + \cdots$$

which depends continuously on φ. We used here the fact that when ψ is pluriharmonic, $\langle \ , \psi \rangle$ depends continuously on ψ. By Theorem 2.11, the above constant is equal to $\langle \ , \varphi \rangle$. Consequently,   is PC.

2) $\Rightarrow$ 1). Let $\mathscr{F}$ be an L^1 bounded family of p.s.h. functions on a neighbourhood of $\mathscr{K}$. We first show that $\langle \ , \varphi \rangle$ is uniformly bounded on $\mathscr{F}$. Since $\langle \ , \Lambda^N(\varphi) \rangle = \langle \ , \varphi \rangle$, we can replace φ with $\Lambda^N(\varphi)$, with N large enough, in order to assume that $\mathscr{F}$ is a bounded family of p.s.h. functions φ on V which are uniformly bounded above. Subtracting from φ a fixed constant allows to assume that these functions are negative. If $\langle \ , \varphi \rangle$ is not uniformly bounded on $\mathscr{F}$, there are φ_n such that $\langle \ , \varphi_n \rangle \leq -n^2$. It follows that the series $\sum n^{-2}\varphi_n$ decreases to a p.s.h. function which is not integrable with respect to   . This contradicts that   is PB.

We deduce that for any neighbourhood W of $\mathcal{K}$, there is a constant $c > 0$ such that $|\langle\ ,\varphi\rangle| \leq c\|\varphi\|_{L^1(W)}$ for φ p.s.h. on W.

We now show that there is a constant $0 < \lambda < 1$ such that $\sup_V \Lambda(\varphi) \leq \lambda$ if φ is a p.s.h. function on V, bounded from above by 1, such that $\langle\ ,\varphi\rangle = 0$. This property implies the last assertion in the proposition. Assume that the property is not satisfied. Then, there are functions φ_n such that $\sup_V \varphi_n = 1$, $\langle\ ,\varphi_n\rangle = 0$ and $\sup_V \Lambda(\varphi_n) \geq 1 - 1/n^2$. By definition of Λ, we have

$$\sup_U \varphi_n \geq \sup_V \Lambda(\varphi_n) \geq 1 - 1/n^2.$$

The submean value inequality for p.s.h. functions implies that φ_n converge to 1 in $L^1_{loc}(V)$. On the other hand, we have

$$1 = |\langle\ ,\varphi_n - 1\rangle| \leq c\|\varphi_n - 1\|_{L^1(W)}.$$

This is a contradiction.

Finally, consider a positive closed $(1,1)$-current S of mass 1 on W. By Proposition A.16, there is a p.s.h. function φ on a neighbourhood of $\overline{U}$ with bounded L^1 norm such that $dd^c\varphi = S$. The submean inequality for p.s.h functions implies that φ is bounded from above by a constant independent of S. We can after subtracting from φ a constant, assume that $\langle\ ,\varphi\rangle = 0$. The p.s.h. functions $\lambda^{-n}\Lambda^n(\varphi)$ are bounded above and satisfy $\langle\ ,\lambda^{-n}\Lambda^n(\varphi)\rangle = \langle\ ,\varphi\rangle = 0$. Hence, they belong to a compact subset of $\mathrm{PSH}(U)$ which is independent of S. If W' is a neighbourhood of $\mathcal{K}$ such that $W' \Subset U$, the mass of $dd^c[\lambda^{-n}\Lambda^n(\varphi)]$ on W' is bounded uniformly on n and on S. Therefore,

$$\|dd^c(f^n)_*(S)\|_{W'} \leq c\lambda^n d_t^n$$

for some constant $c > 0$ independent of n and of S. It follows that $d^*_{k-1} \leq \lambda d_t$. This implies property 1). $\qquad\square$

Theorem 2.34. *Let $f : U \to V$ be a polynomial-like map with large topological degree. Let $\mathscr{P}$ be a bounded family of p.s.h. functions on V. Let K be a compact subset of V such that $f^{-1}(K)$ is contained in the interior of K. Then, the equilibrium measure of f is Hölder continuous on $\mathscr{P}$ with respect to $\mathrm{dist}_{L^1(K)}$. In particular, this measure is moderate.*

Let $\mathrm{DSH}(V)$ denote the space of d.s.h. functions on V, i.e. functions which are differences of p.s.h. functions. They are in particular in $L^p_{loc}(V)$ for every $1 \leq p < +\infty$. Consider on $\mathrm{DSH}(V)$ the following topology: a sequence (φ_n) converges to φ in $\mathrm{DSH}(V)$ if φ_n converge weakly to φ and if we can write $\varphi_n = \varphi_n^+ - \varphi_n^-$ with $\varphi_n^\pm$ in a compact subset of $\mathrm{PSH}(V)$, independent of n. We deduce from the compactness of bounded sets of p.s.h. functions that $\varphi_n \to \varphi$ in all $L^p_{loc}(V)$ with $1 \leq p < +\infty$. Since is PC, it extends by linearity to a continuous functional on $\mathrm{DSH}(V)$.

Proof of Theorem 2.34. Let $\mathscr{P}$ be a compact family of p.s.h. functions on V. We show that is Hölder continuous on $\mathscr{P}$ with respect to $\mathrm{dist}_{L^1(K)}$. We claim that Λ

is Lipschitz with respect to $\mathrm{dist}_{L^1(K)}$. Indeed, if φ, ψ are in $L^1(K)$, we have for the standard volume form Ω on $\mathbb{C}^k$

$$\|\Lambda(\varphi) - \Lambda(\psi)\|_{L^1(K)} = \int_K |\Lambda(\varphi - \psi)|\Omega \leq d_t^{-1} \int_{f^{-1}(K)} |\varphi - \psi| f^*(\Omega).$$

since $f^{-1}(K) \subset K$ and $f^*(\Omega)$ is bounded on $f^{-1}(K)$, this implies that

$$\|\Lambda(\varphi) - \Lambda(\psi)\|_{L^1(K)} \leq \mathrm{const}\|\varphi - \psi\|_{L^1(K)}.$$

Since $\mathscr{P}$ is compact, the functions in $\mathscr{P}$ are uniformly bounded above on U. Therefore, replacing $\mathscr{P}$ by the family of $\Lambda(\varphi)$ with $\varphi \in \mathscr{P}$ allows to assume that functions in $\mathscr{P}$ are uniformly bounded above on V. On the other hand, since   is PC,   is bounded on $\mathscr{P}$. Without loss of generality, we can assume that $\mathscr{P}$ is the set of functions φ such that $\langle \ , \varphi \rangle \geq 0$ and $\varphi \leq 1$. In particular, $\mathscr{P}$ is invariant under Λ. Let $\mathscr{D}$ be the family of d.s.h. functions $\varphi - \Lambda(\varphi)$ with $\varphi \in \mathscr{P}$. This is a compact subset of $\mathrm{DSH}(V)$ which is invariant under Λ, and we have $\langle \ , \varphi' \rangle = 0$ for φ' in $\mathscr{D}$.

Consider a function $\varphi \in \mathscr{P}$. Observe that $\widetilde{\varphi} := \varphi - \langle \ , \varphi \rangle$ is also in $\mathscr{P}$. Define $\widetilde{\Lambda} := \lambda^{-1}\Lambda$ with λ the constant in Theorem 2.33. We deduce from that theorem that $\widetilde{\Lambda}(\widetilde{\varphi})$ is in $\mathscr{P}$. Moreover,

$$\widetilde{\Lambda}\big(\varphi - \Lambda(\varphi)\big) = \widetilde{\Lambda}\big(\widetilde{\varphi} - \Lambda(\widetilde{\varphi})\big) = \widetilde{\Lambda}(\widetilde{\varphi}) - \Lambda\big(\widetilde{\Lambda}(\widetilde{\varphi})\big).$$

Therefore, $\mathscr{D}$ is invariant under $\widetilde{\Lambda}$. This is the key point in the proof. Observe that we can extend $\mathrm{dist}_{L^1(K)}$ to $\mathrm{DSH}(V)$ and that $\widetilde{\Lambda}$ is Lipschitz with respect to this pseudo-distance.

Let ν be a smooth probability measure with support in K. We have seen that $d_t^{-n}(f^n)^*(\nu)$ converge to   . If φ is d.s.h. on V, then

$$\langle d_t^{-n}(f^n)^*(\nu), \varphi \rangle = \langle \nu, \Lambda^n(\varphi) \rangle.$$

Define for φ in $\mathscr{P}$, $\varphi' := \varphi - \Lambda(\varphi)$. We have

$$
\begin{aligned}
\langle \ , \varphi \rangle &= \lim_{n\to\infty} \langle \nu, \Lambda^n(\varphi) \rangle \\
&= \langle \nu, \varphi \rangle - \sum_{n\geq 0} \langle \nu, \Lambda^n(\varphi) \rangle - \langle \nu, \Lambda^{n+1}(\varphi) \rangle \\
&= \langle \nu, \varphi \rangle - \sum_{n\geq 0} \lambda^n \langle \nu, \widetilde{\Lambda}^n(\varphi') \rangle.
\end{aligned}
$$

Since ν is smooth with support in K, it is Lipschitz with respect to $\mathrm{dist}_{L^1(K)}$. We deduce from Lemma 1.19 which is also valid for a pseudo-distance, that the last series defines a Hölder continuous function on $\mathscr{D}$. We use here the invariance of $\mathscr{D}$ under $\widetilde{\Lambda}$. Finally, since the map $\varphi \mapsto \varphi'$ is Lipschitz on $\mathscr{P}$, we conclude that   is Hölder continuous on $\mathscr{P}$ with respect to $\mathrm{dist}_{L^1(K)}$. $\qquad\square$

As in the case of endomorphisms of $\mathbb{P}^k$, we deduce from the above results the following fundamental estimates on the Perron-Frobenius operator Λ.

Corollary 2.35. *Let f be a polynomial-like map with large topological degree as above. Let   be the equilibrium measure and Λ the Perron-Frobenius operator associated to f. Let $\mathscr{D}$ be a bounded subset of d.s.h. functions on V. There are constants $c > 0$, $\delta > 1$ and $\alpha > 0$ such that if ψ is in $\mathscr{D}$, then*

$$\langle\ \ ,e^{\alpha\delta^n|\Lambda^n(\psi)-\langle\ \ ,\psi\rangle|}\rangle \leq c \quad and \quad \|\Lambda^n(\psi)-\langle\ \ ,\psi\rangle\|_{L^q(\ )} \leq cq\delta^{-n}$$

for every $n \geq 0$ and every $1 \leq q < +\infty$.

Corollary 2.36. *Let f, , Λ be as above. Let $0 < v \leq 2$ be a constant. There are constants $c > 0$, $\delta > 1$ and $\alpha > 0$ such that if ψ is a v-Hölder continuous function on V with $\|\psi\|_{\mathscr{C}^v} \leq 1$, then*

$$\langle\ \ ,e^{\alpha\delta^{nv/2}|\Lambda^n(\psi)-\langle\ \ ,\psi\rangle|}\rangle \leq c \quad and \quad \|\Lambda^n(\psi)-\langle\ \ ,\psi\rangle\|_{L^q(\ )} \leq cq^{v/2}\delta^{-nv/2}$$

for every $n \geq 0$ and every $1 \leq q < +\infty$. Moreover, δ is independent of v.

The following results are deduced as in the case of endomorphisms of $\mathbb{P}^k$.

Theorem 2.37. *Let $f : U \to V$ be a polynomial-like map with large topological degree and   the equilibrium measure of f. Then f is exponentially mixing. More precisely, there is a constant $0 < \lambda < 1$, such that if $1 < p \leq +\infty$, we have*

$$|\langle\ \ ,(\varphi \circ f^n)\psi\rangle - \langle\ \ ,\varphi\rangle\langle\ \ ,\psi\rangle| \leq c_p\lambda^n\|\varphi\|_{L^p(\ )}\|\psi\|_{L^1(V)}$$

for φ in $L^p(\ )$, ψ p.s.h. on V and $n \geq 0$, where $c_p > 0$ is a constant independent of φ, ψ. If v is such that $0 \leq v \leq 2$, then there is a constant $c_{p,v} > 0$ such that

$$|\langle\ \ ,(\varphi \circ f^n)\psi\rangle - \langle\ \ ,\varphi\rangle\langle\ \ ,\psi\rangle| \leq c_{p,v}\lambda^{nv/2}\|\varphi\|_{L^p(\ )}\|\psi\|_{\mathscr{C}^v}$$

for φ in $L^p(\ )$, ψ a $\mathscr{C}^v$ function on V and $n \geq 0$.

The following result gives the exponential mixing of any order. It can be extended to Hölder continuous observables using the theory of interpolation between Banach spaces.

Theorem 2.38. *Let f,   be as in Theorem 2.37 and $r \geq 1$ an integer. Then there are constants $c > 0$ and $0 < \lambda < 1$ such that*

$$\left|\langle\ \ ,\psi_0(\psi_1 \circ f^{n_1})\dots(\psi_r \circ f^{n_r})\rangle - \prod_{i=0}^{r}\langle\ \ ,\psi_i\rangle\right| \leq c\lambda^n\prod_{i=0}^{r}\|\psi_i\|_{L^1(V)}$$

for $0 = n_0 \leq n_1 \leq \cdots \leq n_r$, $n := \min_{0 \leq i < r}(n_{i+1} - n_i)$ and ψ_i p.s.h. on V.

As in Section 1, we deduce the following result, as a consequence of Gordin's theorem and of the exponential decay of correlations.

Theorem 2.39. *Let f be a polynomial-like map with large topological degree as above. Let φ be a test function which is $\mathscr{C}^\nu$ with $\nu > 0$, or is d.s.h. on V. Then, either φ is a coboundary or it satisfies the central limit theorem with the variance $\sigma > 0$ given by*

$$\sigma^2 := \langle\ ,\varphi^2\rangle + 2\sum_{n\geq 1}\langle\ ,\varphi(\varphi\circ f^n)\rangle.$$

The following result is obtained as in Theorem 1.102, as a consequence of the exponential estimates in Corollaries 2.35 and 2.36.

Theorem 2.40. *Let f be a polynomial-like map with large topological degree as above. Then, the equilibrium measure   of f satisfies the weak large deviations theorem for bounded d.s.h. observables and for Hölder continuous observables. More precisely, if a function ψ is bounded d.s.h. or Hölder continuous then for every $\varepsilon > 0$ there is a constant $h_\varepsilon > 0$ such that*

$$\left\{z \in \mathrm{supp}(\ ) : \left|\frac{1}{N}\sum_{n=0}^{N-1}\varphi\circ f^n(z) - \langle\ ,\varphi\rangle\right| > \varepsilon\right\} \leq e^{-N(\log N)^{-2}h_\varepsilon}$$

for all N large enough.

Theorem 1.91 can be extended to polynomial-like maps with large topological degrees.

Theorem 2.41. *Let $f : U \to V$ be a polynomial-like map with large topological degree and   its equilibrium measure. Let φ be an observable on V with values in $\mathbb{R}\cup\{-\infty\}$ such that e^φ is Hölder continuous, $H := \{\varphi = -\infty\}$ is an analytic subset of V and $|\varphi| \lesssim |\log\mathrm{dist}(\cdot,H)|^\rho$ near H for some $\rho > 0$. If $\langle\ ,\varphi\rangle = 0$ and φ is not a coboundary, then the almost sure invariance principle holds for φ. In particular, the almost sure central limit theorem holds for such observables.*

A technical point here is to prove that if H_t is the t-neighbourhood of an analytic set H, then $(H_t) \lesssim t^\alpha$ for $t > 0$ and for some constant $\alpha > 0$. This property is a consequence of the fact that   is moderate.

We also have the following version of Theorem 1.92. In the proof, one uses Proposition 2.21 instead of Proposition 1.51.

Theorem 2.42. *Let $f : U \to V$ be a polynomial-like map with large topological degree $d_t > 1$ and   its equilibrium measure. Then $(U,\ ,f)$ is measurably conjugate to a one-sided d_t-shift.*

Exercise 2.43. Assume that for every positive closed $(1,1)$-current S on V we have $\limsup\|(f^n)_*(S)\|^{1/n} < d_t$. Show that   is PB and deduce that $d_{k-1}^* < d_t$. Hint: write $S = dd^c\varphi$.

Exercise 2.44. Let ν be a positive measure with compact support in $\mathbb{C}$. Prove that ν is moderate if and only if there are positive constants α and c such that for every disc D of radius r, $\nu(D) \leq cr^\alpha$. Give an example showing that this condition is not sufficient in $\mathbb{C}^2$.

2.5 Holomorphic Families of Maps

In this paragraph, we consider polynomial-like maps $f_s : U_s \to V_s$ depending holomorphically on a parameter $s \in \Sigma$. We will show that the Green measure $_s$ of f_s depends "holomorphically" on s and then we study the dependence of the Lyapounov exponents on the parameters. Since the problems are local, we assume for simplicity that Σ is a ball in $\mathbb{C}^l$. Of course, we assume that U_s and V_s depend continuously on s. Observe that if we replace V_s with a convex open set $V'_s \subset V_s$ and U_s by $f_s^{-1}(V'_s)$ with $V_s \setminus V'_s$ small enough, the map f_s is still polynomial-like. So, for simplicity, assume that $V := V_s$ is independent of s. Let $U_\Sigma := \cup_s \{s\} \times U_s$. This is an open set in $V_\Sigma := \Sigma \times V$. Define the holomorphic map $F : U_\Sigma \to V_\Sigma$ by $F(s,z) := (s, f_s(z))$. This map is proper. By continuity, the topological degree d_t of f_s is independent of s. So, the topological degree of F is also d_t. Define $\mathscr{K}_\Sigma := \cap_{n \geq 0} F^{-n}(V_\Sigma)$. Then $\mathscr{K}_\Sigma$ is closed in U_Σ. If $\pi : \Sigma \times \mathbb{C}^k \to \Sigma$ is the canonical projection, then π is proper on $\mathscr{K}_\Sigma$ and $\mathscr{K}_s := \mathscr{K}_\Sigma \cap \pi^{-1}(s)$ is the filled Julia set of f_s.

It is not difficult to show that $\mathscr{K}_s$ depends upper semi-continuously on s with respect to the Hausdorff metric on compact sets of V. This means that if W_{s_0} is a neighbourhood of $\mathscr{K}_{s_0}$, then $\mathscr{K}_s$ is contained in W_{s_0} for s closed enough to s_0. We will see that for maps with large topological degree, $s \mapsto {}_s$ is continuous in a strong sense. However, in general, the Julia set $\mathscr{J}_s$, i.e. the support of the equilibrium measure $_s$, does not depend continuously on s.

In our context, the goal is to construct and to study currents which measure the bifurcation, i.e. the discontinuity of $s \mapsto \mathscr{J}_s$. We have the following result due to Pham [PH].

Proposition 2.45. *Let* $(f_s)_{s \in \Sigma}$ *be as above. Then, there is a positive closed current* $\mathscr{R}$ *of bidegree* (k,k), *supported on* $\mathscr{K}_\Sigma$ *such that the slice* $\langle \mathscr{R}, \pi, s \rangle$ *is equal to the equilibrium measure* $_s$ *of* f_s *for* $s \in \Sigma$. *Moreover, if* φ *is a p.s.h. function on a neighbourhood of* $\mathscr{K}_\Sigma$, *then the function* $s \mapsto \langle {}_s, \varphi(s, \cdot) \rangle$ *is either equal to* $-\infty$ *or is p.s.h. on* Σ.

Proof. Let Ω be a smooth probability measure with compact support in V. Define the positive closed (k,k)-current Θ on $\Sigma \times V$ by $\Theta := \tau^*(\Omega)$ where $\tau : \Sigma \times V \to V$ is the canonical projection. Observe that the slice $\langle \Theta, \pi, s \rangle$ coincides with Ω on $\{s\} \times V$, since Ω is smooth. Define $\Theta_n := d_t^{-n}(F^n)^*(\Theta)$. The slice $\langle \Theta_n, \pi, s \rangle$ can be identified with $d_t^{-n}(f_s^n)^*(\Omega)$ on $\{s\} \times V$. This is a smooth probability measure which tends to $_s$ when n goes to infinity.

Since the problem is local for s, we can assume that all the forms Θ_n are supported on $\Sigma \times K$ for some compact subset K of V. As we mentioned in Appendix A.3, since these forms have slice mass 1, they belong to a compact family of currents. Therefore, we can extract a sequence Θ_{n_i} which converges to some current $\mathscr{R}$ with slice mass 1. We want to prove that $\langle \mathscr{R}, \pi, s \rangle = {}_s$.

Let φ be a smooth p.s.h. function on a neighbourhood of $\mathscr{K}_\Sigma$. So, for n large enough, φ is defined on the support of Θ_n (we reduce Σ if necessary). By slicing theory, $\pi_*(\Theta_{n_i} \wedge \varphi)$ is equal to the p.s.h. function $\psi_{n_i}(s) := \langle \Theta_{n_i}, \pi, s \rangle(\varphi)$ and $\pi_*(\mathscr{R} \wedge \varphi)$ is equal to the p.s.h. function $\psi(s) := \langle \mathscr{R}, \pi, s \rangle(\varphi)$ in the sense of

currents. By definition of $\mathscr{R}$, since π_* is continuous on currents supported on $\Sigma \times K$, ψ_{n_i} converge to ψ in $L^1_{loc}(\Sigma)$. On the other hand, $\langle \Theta_{n_i}, \pi, s \rangle$ converge to $_s$. So, the function $\psi'(s) := \lim \psi_{n_i}(s) = \langle {}_s, \varphi \rangle$ is equal to $\psi(s)$ almost everywhere. Since ψ_{n_i} and ψ are p.s.h., the Hartogs' lemma implies that $\psi' \leq \psi$. We show the inequality $\psi'(s) \geq \psi(s)$.

The function ψ is p.s.h., hence it is strongly upper semi-continuous. Therefore, there is a sequence (s_n) converging to s such that $\psi'(s_n) = \psi(s_n)$ and $\psi(s_n)$ converge to $\psi(s)$. Up to extracting a subsequence, we can assume that $_{s_n}$ converge to some probability measure $_s'$. By continuity, $_s'$ is totally invariant under f_s. We deduce from Proposition 2.13 that $\langle {}_s', \varphi(s, \cdot) \rangle \leq \langle {}_s, \varphi(s, \cdot) \rangle$. The first integral is equal to $\psi(s)$, the second one is equal to $\psi'(s)$. Therefore, $\psi(s) \leq \psi'(s)$. The identity $\langle \mathscr{R}, \pi, s \rangle = {}_s$ follows.

The second assertion in the proposition is also a consequence of the above arguments. This is clear when φ is smooth. The general case is deduced using an approximation of φ by a decreasing sequence of smooth p.s.h. functions.   □

Let $\mathrm{Jac}(F)$ denote the Jacobian of F with respect to the standard volume form on $\Sigma \times \mathbb{C}^k$. Its restriction to $\pi^{-1}(s)$ is the Jacobian $\mathrm{Jac}(f_s)$ of f_s. Since $\mathrm{Jac}(F)$ is a p.s.h. function, we can apply the previous proposition and deduce that the function $L_k(s) := \frac{1}{2} \langle {}_s, \log \mathrm{Jac}(f_s) \rangle$ is p.s.h. on Σ. Indeed, by Theorem 2.16, this function is bounded from below by $\frac{1}{2} \log d_t$, hence it is not equal to $-\infty$. By Oseledec's theorem 1.119, $L_k(s)$ is the sum of the Lyapounov exponents of f_s. We deduce the following result of [DS1].

Corollary 2.46. *Let $(f_s)_{s \in \Sigma}$ be as above. Then, the sum of the Lyapounov exponents associated to the equilibrium measure $_s$ of f_s is a p.s.h. function on s. In particular, it is upper semi-continuous.*

Pham defined in [PH] *the bifurcation (p,p)-currents* by $\mathscr{B}^p := (dd^c L_k)^p$ for $1 \leq p \leq \dim \Sigma$. The wedge-product is well-defined since L_k is locally bounded: it is bounded from below by $\frac{1}{2} \log d_t$. Very likely, these currents play a crucial role in the study of bifurcation as we see in the following observation. Assume that the critical set of f_{s_0} does not intersect the filled Julia set $\mathscr{K}_{s_0}$ for some $s_0 \in \Sigma$. Since the filled Julia sets $\mathscr{K}_s$ vary upper semi-continuously in the Hausdorff metric, $\log \mathrm{Jac}(F)$ is pluriharmonic near $\{s_0\} \times \mathscr{K}_{s_0}$. It follows that L_k is pluriharmonic and $\mathscr{B}^p = 0$ in a neighbourhood of s_0[8]. On the other hand, using Kobayashi metric, it is easy to show that f is uniformly hyperbolic on $\mathscr{K}_s$ for s close to s_0. It follows that $\mathscr{K}_s = \mathscr{J}_s$ and $s \mapsto \mathscr{J}_s$ is continuous near s_0, see [FS6].

Note that L_k is equal in the sense of currents to $\pi_*(\log \mathrm{Jac}(F) \wedge \mathscr{R})$, where $\mathscr{R}$ is the current in Proposition 2.45. Therefore, $\mathscr{B}$ can be obtained using the formula

$$\mathscr{B} = dd^c \pi_*(\log \mathrm{Jac}(F) \wedge \mathscr{R}) = \pi_*([\mathscr{C}_F] \wedge \mathscr{R}),$$

since $dd^c \log |\mathrm{Jac}(F)| = [\mathscr{C}_F]$, the current of integration on the critical set $\mathscr{C}_F$ of F. We also have the following property of the function L_k.

---

[8] This observation was made by the second author for the family $z^2 + c$, with $c \in \mathbb{C}$. He showed that the bifurcation measure is the harmonic measure associated to the Mandelbrot set [SI].

Theorem 2.47. *Let $(f_s)_{s \in \Sigma}$ be a family of polynomial-like maps as above. Assume that f_{s_0} has a large topological degree for some $s_0 \in \Sigma$. Then L_k is Hölder continuous in a neighbourhood of s_0. In particular, the bifurcation currents $\mathscr{B}^p$ are moderate for $1 \le p \le \dim\Sigma$.*

Let Λ_s denote the Perron-Frobenius operator associated to f_s. For any Borel set B, denote by Ω_B the standard volume form on $\mathbb{C}^k$ restricted to B. We first prove some preliminary results.

Lemma 2.48. *Let W be a neighbourhood of the filled Julia set $\mathscr{K}_{s_0}$ of f_{s_0}. Then, there is a neighbourhood Σ_0 of s_0 such that $\langle \ _s, \varphi \rangle$ depends continuously on (s, φ) in $\Sigma_0 \times \mathrm{PSH}(W)$.*

Proof. We first replace Σ with a neighbourhood Σ_0 of s_0 small enough. So, for every $s \in \Sigma$, the filled Julia set of f_s is contained in $U := f_{s_0}^{-1}(V)$ and in W. We also reduce the size of V in order to assume that f_s is polynomial-like on a neighbourhood of $\overline{U}$ with values in a neighbourhood V' of $\overline{V}$. Moreover, since $\langle \ _s, \varphi \rangle = \langle \ _s, \Lambda_s(\varphi) \rangle$ and $\Lambda_s(\varphi)$ depends continuously on (s, φ) in $\Sigma \times \mathrm{PSH}(W)$, we can replace φ with $\Lambda_s^N(\varphi)$ with N large enough and $s \in \Sigma$, in order to assume that $W = V$. Finally, since $\Lambda_s(\varphi)$ is defined on V', it is enough to prove the continuity for φ p.s.h. on V such that $\varphi \le 1$ and $\langle \Omega_U, \varphi \rangle \ge 0$. Denote by $\mathscr{P}$ the family of such functions φ. Since $\ _{s_0}$ is PC, we have $|\langle \ _{s_0}, \varphi \rangle| \le A$ for some constant $A \ge 1$ and for $\varphi \in \mathscr{P}$. Let $\mathscr{P}'$ denote the family of p.s.h. functions ψ such that $\psi \le 2A$ and $\langle \ _{s_0}, \psi \rangle = 0$. The function $\varphi' := \varphi - \langle \ _{s_0}, \varphi \rangle$ belongs to this family. Observe that $\mathscr{P}'$ is bounded and therefore if $A' \ge 1$ is a fixed constant large enough, we have $|\langle \Omega_U, \psi \rangle| \le A'$ for $\psi \in \mathscr{P}'$.

Fix an integer N large enough. By Theorem 2.33, $\Lambda_{s_0}^N(\varphi') \le 1/8$ on V' and $|\langle \Omega_U, \Lambda_{s_0}^N(\varphi') \rangle| \le 1/8$ for φ' as above. We deduce that $2\Lambda_{s_0}^N(\varphi') - \langle \Omega_U, 2\Lambda_{s_0}^N(\varphi') \rangle$ is a function in $\mathscr{P}$, smaller than $1/2$ on V'. This function differs from $2\Lambda_{s_0}^N(\varphi)$ by a constant. So, it is equal to $2\Lambda_{s_0}^N(\varphi) - \langle \Omega_U, 2\Lambda_{s_0}^N(\varphi) \rangle$. When Σ_0 is small enough, by continuity, the operator $L_s(\varphi) := 2\Lambda_s^N(\varphi) - \langle \Omega_U, 2\Lambda_s^N(\varphi) \rangle$ preserves $\mathscr{P}$ for $s \in \Sigma_0$. Therefore, since Λ_s preserves constant functions, we have

$$\Lambda_s^{mN}(\varphi) = \Lambda_s^{(m-1)N}\left[\langle \Omega_U, \Lambda_s^N(\varphi) \rangle + 2^{-1}L_s(\varphi)\right]$$
$$= \langle \Omega_U, \Lambda_s^N(\varphi) \rangle + 2^{-1}\Lambda_s^{(m-1)N}(L_s(\varphi)).$$

By induction, we obtain

$$\Lambda_s^{mN}(\varphi) = \langle \Omega_U, \Lambda_s^N(\varphi) \rangle + \cdots + 2^{-m+1}\langle \Omega_U, \Lambda_s^N(L_s^{m-1}(\varphi)) \rangle + 2^{-m}L_s^m(\varphi)$$
$$= \langle \Omega_U, \Lambda_s^N\left[\varphi + \cdots + 2^{-m+1}L_s^{m-1}(\varphi)\right] \rangle + 2^{-m}L_s^m(\varphi)$$
$$= \langle d_t^{-N}(f_s^N)^*(\Omega_U), \varphi + \cdots + 2^{-m+1}L_s^{m-1}(\varphi) \rangle + 2^{-m}L_s^m(\varphi).$$

We deduce from the above property of L_s that the last term converges uniformly to 0 when m goes to infinity. The sum in the first term converges normally to the p.s.h. function $\sum_{m \ge 1} 2^{-m+1}L_s^{m-1}(\varphi)$, which depends continuously on (s, φ). Therefore, $\Lambda_s^{mN}(\varphi)$ converge to a constant which depends continuously on (s, φ). But we know that the limit is $\langle \ _s, \varphi \rangle$. The lemma follows. $\qquad\square$

Using the same approach as in Theorem 2.34, we prove the following result.

Theorem 2.49. *Let f_s, s_0 and W be as in Theorem 2.47 and Lemma 2.48. Let K be a compact subset of W such that $f_{s_0}^{-1}(K)$ is contained in the interior of K. There is a neighbourhood Σ_0 of s_0 such that if $\mathscr{P}$ is a bounded family of p.s.h. functions on W, then $(s, \varphi) \mapsto \langle \ _s, \varphi \rangle$ is Hölder continuous on $\Sigma_0 \times \mathscr{P}$ with respect to the pseudo-distance $\mathrm{dist}_{L^1(K)}$ on $\mathscr{P}$.*

Proof. We replace Σ with Σ_0 as in Lemma 2.48. It is not difficult to check that $(s, \varphi) \mapsto (s, \Lambda_s(\varphi))$ is locally Lipschitz with respect to $\mathrm{dist}_{L^1(K)}$. So, replacing (s, φ) by $(s, \Lambda_s^N(\varphi))$ with N large enough allows to assume that $W = V$. Let $\widehat{\mathscr{P}}$ be the set of (s, φ) in $\Sigma \times \mathrm{PSH}(V)$ such that $\varphi \leq 1$ and $\langle \ _s, \varphi \rangle \geq 0$. By Lemma 2.48, such functions φ belong to a compact subset of $\mathrm{PSH}(V)$. It is enough to prove that $(s, \varphi) \mapsto \langle \ _s, \varphi \rangle$ is Hölder continuous on $\widehat{\mathscr{P}}$.

Let $\widehat{\mathscr{D}}$ denote the set of $(s, \varphi - \Lambda_s(\varphi))$ with $(s, \varphi) \in \widehat{\mathscr{P}}$. Consider the operator $\widehat{\Lambda}(s, \psi) := (s, \lambda^{-1} \Lambda_s(\psi))$ on $\mathscr{D}$ as in Theorem 2.34 where $\lambda < 1$ is a fixed constant close enough to 1. Theorem 2.33 and the continuity in Lemma 2.48 imply that $\widehat{\Lambda}$ preserves $\widehat{\mathscr{D}}$. Therefore, we only have to follow the arguments in Theorem 2.34. $\square$

Proof of Theorem 2.47. We replace Σ with a small neighbourhood of s_0. Observe that $\log \mathrm{Jac}(f_s)$, $s \in \Sigma$, is a bounded family of p.s.h. functions on U. By Theorem 2.49, it is enough to show that $s \mapsto \log \mathrm{Jac}(f_s)$ is Hölder continuous with respect to $\mathrm{dist}_{L^1(K)}$.

We also deduce from Theorem A.22 that $\langle \Omega_K, e^{\lambda |\log \mathrm{Jac}(f_s)|} \rangle \leq A$ for some positive constants λ and A. Reducing V and Σ allows to assume that $\mathrm{Jac}(F)$, their derivatives and the vanishing order of $\mathrm{Jac}(F)$ are bounded on $\Sigma \times U$ by some constant m.

Fix a constant $\alpha > 0$ small enough and a constant $A > 0$ large enough. Define $\psi(s) := \langle \Omega_K, \log \mathrm{Jac}(f_s) \rangle$. Consider s and t in Σ such that $r := \|s - t\|$ is smaller than a fixed small constant. We will compare $|\psi(s) - \psi(t)|$ with $r^{\lambda \alpha}$ in order to show that ψ is Hölder continuous with exponent $\lambda \alpha$. Define $S := \{z \in U, \ \mathrm{Jac}(f_s) < 2r^{2\alpha}\}$. We will bound separately

$$\langle \Omega_{K \setminus S}, \log \mathrm{Jac}(f_s) - \log \mathrm{Jac}(f_t) \rangle$$

and

$$\langle \Omega_{K \cap S}, \log \mathrm{Jac}(f_s) - \log \mathrm{Jac}(f_t) \rangle.$$

Note that $\psi(s) - \psi(t)$ is the sum of the above two integrals.

Consider now the integral on $K \setminus S$. The following estimates are only valid on $K \setminus S$. Since the derivatives of $\mathrm{Jac}(F)$ is bounded, we have $\mathrm{Jac}(f_t) \geq r^{2\alpha}$. It follows that the derivatives on t of $\log \mathrm{Jac}(f_t)$ is bounded by $Ar^{-2\alpha}$. We deduce that

$$|\log \mathrm{Jac}(f_s) - \log \mathrm{Jac}(f_t)| \leq Ar^{1-2\alpha}.$$

Therefore,

$$|\langle \Omega_{K \setminus S}, \log \mathrm{Jac}(f_s) - \log \mathrm{Jac}(f_t) \rangle| \leq \|\Omega_{K \setminus S}\| Ar^{1-2\alpha} \leq r^{\lambda \alpha}.$$

We now estimate the integral on $K \cap S$. Its absolute value is bounded by

$$\langle \Omega_{K \cap S}, |\log \mathrm{Jac}(f_s)| \rangle + \langle \Omega_{K \cap S}, |\log \mathrm{Jac}(f_t)| \rangle.$$

We deduce from the estimate $\langle \Omega_K, e^{\lambda |\log \mathrm{Jac}(f_s)|} \rangle \leq A$ that $\mathrm{volume}(K \cap S) \leq A r^{2\lambda \alpha}$. Therefore, by Cauchy-Schwarz's inequality, we have

$$\langle \Omega_{K \cap S}, |\log \mathrm{Jac}(f_s)| \rangle \lesssim \mathrm{volume}(K \cap S)^{1/2} \langle \Omega_K, |\log \mathrm{Jac}(f_s)|^2 \rangle^{1/2}$$
$$\lesssim r^{\lambda \alpha} \langle \Omega_K, e^{\lambda |\log \mathrm{Jac}(f_s)|} \rangle^{1/2} \lesssim r^{\lambda \alpha}.$$

The estimate holds for f_t instead of f_s. Hence, ψ is Hölder continuous. The fact that $\mathscr{B}^p$ are moderate follows from Theorem A.34. $\qquad \square$

The following result of Pham generalizes Corollary 2.46 and allows to define other bifurcation currents by considering $dd^c L_p$ or their wedge-products [PH].

Theorem 2.50. *Let $(f_s)_{s \in \Sigma}$ be a holomorphic family of polynomial-like maps as above. Let $\chi_1(s) \geq \cdots \geq \chi_k(s)$ be the Lyapounov exponents of the equilibrium measure μ_s of f_s. Then, for $1 \leq p \leq k$, the function*

$$L_p(s) := \chi_1(s) + \cdots + \chi_p(s)$$

is p.s.h. on Σ. In particular, L_p is upper semi-continuous.

Proof. Observe that $L_p(s) \geq \frac{p}{k} L_k(s) \geq \frac{p}{2k} \log d_t$. We identify the tangent space of V at any point with $\mathbb{C}^k$. So, the differential $Df_s(z)$ of f_s at a point $z \in U_s$ is a linear self-map on $\mathbb{C}^k$ which depends holomorphically on (s, z). It induces a linear self-map on the exterior product $\bigwedge^p \mathbb{C}^k$ that we denote by $D^p f_s(z)$. This map depends holomorphically on (s, z). In the standard coordinate system on $\bigwedge^p \mathbb{C}^k$, the function $(s, z) \mapsto \log \|D^p f_s(z)\|$ is p.s.h. on U_Σ. By Proposition 2.45, the function $\psi_1(s) := \langle \mu_s, \log \|D^p f_s\| \rangle$ is p.s.h. or equal to $-\infty$ on Σ. Define in the same way the functions $\psi_n(s) := \langle \mu_s, \log \|D^p f_s^n\| \rangle$ associated to the iterate f_s^n of f_s. We have

$$D^p f_s^{n+m}(z) = D^p f_s^m(f_s^n(z)) \circ D^p f_s^n(z).$$

Hence,

$$\|D^p f_s^{n+m}(z)\| \leq \|D^p f_s^m(f_s^n(z))\| \, \|D^p f_s^n(z)\|.$$

We deduce using the invariance of μ_s that

$$\psi_{m+n}(s) \leq \psi_m(s) + \psi_n(s).$$

Therefore, the sequence $n^{-1} \psi_n$ decreases to $\inf_n n^{-1} \psi_n$. So, the limit is p.s.h. or equal to $-\infty$. On the other hand, Oseledec's theorem 1.119 implies that the limit is equal to $L_p(s)$ which is a positive function. It follows that $L_p(s)$ is p.s.h. $\qquad \square$

Consider now the family f_s of endomorphisms of algebraic degree $d \geq 2$ of $\mathbb{P}^k$ with $s \in \mathscr{H}_d(\mathbb{P}^k)$. We can lift f_s to polynomial-like maps on $\mathbb{C}^{k+1}$ and apply the above results. The construction of the bifurcation currents $\mathscr{B}^p$ can be

obtained directly using the Green measures of f_s. This was done by Bassanelli-Berteloot in [BB]. They studied some properties of the bifurcation currents and obtained nice formulas for that currents in terms of the Green functions. We also refer to DeMarco, Dujardin-Favre, McMullen, Milnor, Sibony and Silverman [DM, DM1, DF, MM, MI1, SI, SJ] for results in dimension one.

Exercise 2.51. If f is an endomorphism in $\mathscr{H}_d(\mathbb{P}^k)$, denote by $L_k(f)$ the sum of the Lyapounov exponents of the equilibrium measure. Show that $f \mapsto L_k(f)$ is locally Hölder continuous on $\mathscr{H}_d(\mathbb{P}^k)$. Deduce that the bifurcation currents are moderate. Hint: use that the lift of f to $\mathbb{C}^{k+1}$ has always a Lyapounov exponent equal to $\log d$.

Exercise 2.52. Find a family $(f_s)_{s \in \Sigma}$ such that $\mathscr{J}_s$ does not vary continuously.

Exercise 2.53. A family $(X_s)_{s \in \Sigma}$ of compact subsets in V is lower semi-continuous at s_0 if for every $\varepsilon > 0$, X_{s_0} is contained in the ε-neighbourhood of X_s when s is close enough to s_0. If $(\nu_s)_{s \in \Sigma}$ is a continuous family of probability measures on V, show that $s \mapsto \text{supp}(\nu_s)$ is lower semi-continuous. If $(f_s)_{s \in \Sigma}$ is a holomorphic family of polynomial-like maps, deduce that $s \mapsto \mathscr{J}_s$ is lower semi-continuous. Show that if $\mathscr{J}_{s_0} = \mathscr{K}_{s_0}$, then $s \mapsto \mathscr{J}_s$ is continuous at s_0 for the Hausdorff metric.

Exercise 2.54. Assume that f_{s_0} is of large topological degree. Let $\delta > 0$ be a constant small enough. Using the continuity of $s \mapsto {}_s$, show that if p_{s_0} is a repelling fixed point in $\mathscr{J}_{s_0}$ for f_{s_0}, there are repelling fixed points p_s in $\mathscr{J}_s$ for f_s, with $|s - s_0| < \delta$, such that $s \mapsto p_s$ is holomorphic. Suppose $s \mapsto \mathscr{J}_s$ is continuous with respect to the Hausdorff metric. Construct a positive closed current $\mathscr{R}$ supported on $\cup_{|s-s_0|<\delta} \{s\} \times \mathscr{J}_s$ with slices ${}_s$. Deduce that if $\mathscr{J}_{s_0}$ does not contain critical points of f_{s_0} then $s \mapsto L_k(s)$ is pluriharmonic near s_0.

Notes. Several results in this section still hold for larger classes of polynomial-like maps. For example, the construction of the equilibrium measure is valid for a manifold V admitting a smooth strictly p.s.h. function. The dd^c-method was originally introduced for polynomial-like maps. However, we have seen that it is also effective for endomorphisms of $\mathbb{P}^k$. In a forthcoming survey, we will show that the method can be extended to other dynamical systems. Several statistical properties obtained in this section are new.

Appendix: Currents and Pluripotential Theory

In this appendix, we recall some basic notions and results on complex geometry and on currents in the complex setting. Most of the results are classical and their proofs are not given here. In constrast, we describe in detail some notions in order to help the reader who are not familiar with complex geometry or currents. The main references for the abstract theory of currents are [CH, DR, FE, SC, WA]. The reader will find in [DEM, GU, HO, LE, N] the basics on currents on complex manifolds. We also refer to [DEM, GH, HB, VO] for the theory of compact Kähler manifolds.

A.1 Projective Spaces and Analytic Sets

In this paragraph, we recall the definition of complex projective spaces. We then discuss briefly compact Kähler manifolds, projective manifolds and analytic sets.

The complex projective space $\mathbb{P}^k$ is a compact complex manifold of dimension k. It is obtained as the quotient of $\mathbb{C}^{k+1} \setminus \{0\}$ by the natural multiplicative action of $\mathbb{C}^*$. In other words, $\mathbb{P}^k$ is the parameter space of the complex lines passing through 0 in $\mathbb{C}^{k+1}$. The image of a subspace of dimension $p+1$ of $\mathbb{C}^{k+1}$ is a submanifold of dimension p in $\mathbb{P}^k$, bi-holomorphic to $\mathbb{P}^p$, and is called *a projective subspace of dimension p. Hyperplanes* of $\mathbb{P}^k$ are projective subspaces of dimension $k-1$. The group $\mathrm{GL}(\mathbb{C}, k+1)$ of invertible linear endomorphisms of $\mathbb{C}^{k+1}$ induces the group $\mathrm{PGL}(\mathbb{C}, k+1)$ of automorphisms of $\mathbb{P}^k$. It acts transitively on $\mathbb{P}^k$ and sends projective subspaces to projective subspaces.

Let $z = (z_0, \ldots, z_k)$ denote the standard coordinates of $\mathbb{C}^{k+1}$. Consider the equivalence relation: $z \sim z'$ *if there is* $\lambda \in \mathbb{C}^*$ *such that* $z = \lambda z'$. The projective space $\mathbb{P}^k$ is the quotient of $\mathbb{C}^{k+1} \setminus \{0\}$ by this relation. We can cover $\mathbb{P}^k$ by open sets U_i associated to the open sets $\{z_i \neq 0\}$ in $\mathbb{C}^{k+1} \setminus \{0\}$. Each U_i is bi-holomorphic to $\mathbb{C}^k$ and $(z_0/z_i, \ldots, z_{i-1}/z_i, z_{i+1}/z_i, \ldots, z_k/z_i)$ is a coordinate system on this chart. The complement of U_i is the hyperplane defined by $\{z_i = 0\}$. So, $\mathbb{P}^k$ can be considered as a natural compactification of $\mathbb{C}^k$. We denote by $[z_0 : \cdots : z_k]$ the point of $\mathbb{P}^k$ associated to $(z_0, \ldots, z_k)$. This expression is *the homogeneous coordinates* on $\mathbb{P}^k$. Projective spaces are compact Kähler manifolds. We will describe this notion later.

Let X be a complex manifold of dimension k. Let φ be a differential l-form on X. In local holomorphic coordinates $z = (z_1, \ldots, z_k)$, it can be written as

$$\varphi(z) = \sum_{|I|+|J|=l} \varphi_{IJ} dz_I \wedge d\bar{z}_J,$$

where φ_{IJ} are complex-valued functions, $dz_I := dz_{i_1} \wedge \ldots \wedge dz_{i_p}$ if $I = (i_1, \ldots, i_p)$, and $d\bar{z}_J := d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}$ if $J = (j_1, \ldots, j_q)$. The *conjugate* of φ is

$$\overline{\varphi}(z) := \sum_{|I|+|J|=l} \overline{\varphi}_{IJ} d\bar{z}_I \wedge dz_J.$$

The form φ is real if and only if $\varphi = \overline{\varphi}$.

We say that φ is a form of *of bidegree* (p, q) if $\varphi_{IJ} = 0$ when $(|I|, |J|) \neq (p, q)$. The bidegree does not depend on the choice of local coordinates. Let $T_X^{\mathbb{C}}$ denote the complexification of the tangent bundle of X. The complex structure on X induces a linear endomorphism $\mathscr{J}$ on the fibers of $T_X^{\mathbb{C}}$ such that $\mathscr{J}^2 = -\mathrm{id}$. This endomorphism induces a decomposition of $T_X^{\mathbb{C}}$ into the direct sum of two proper sub-bundles of dimension k: *the holomorphic part* $T_X^{1,0}$ associated to the eigenvalue i of $\mathscr{J}$, and the *anti-holomorphic part* $T_X^{0,1}$ associated to the eigenvalue $-i$. Let $\Omega_X^{1,0}$ and $\Omega_X^{0,1}$ denote the dual bundles of $T_X^{1,0}$ and $T_X^{0,1}$. Then, (p, q)-forms sections of the vector bundle $\bigwedge^p \Omega^{1,0} \otimes \bigwedge^q \Omega^{0,1}$.

If φ is a (p,q)-form then the differential $d\varphi$ is the sum of a $(p+1,q)$-form and a $(p,q+1)$-form. We then denote by $\partial\varphi$ the part of bidegree $(p+1,q)$ and $\overline{\partial}\varphi$ the the part of bidegree $(p,q+1)$. The operators ∂ and $\overline{\partial}$ extend linearly to arbitrary forms φ. The operator d is real, i.e. it sends real forms to real forms but ∂ and $\overline{\partial}$ are not real. The identity $d \circ d = 0$ implies that $\partial \circ \partial = 0$, $\overline{\partial} \circ \overline{\partial} = 0$ and $\partial\overline{\partial} + \overline{\partial}\partial = 0$. Define $d^c := \frac{\sqrt{-1}}{2\pi}(\overline{\partial} - \partial)$. This operator is real and satisfies $dd^c = \frac{\sqrt{-1}}{\pi}\partial\overline{\partial}$. Note that the above operators commute with the pull-back by holomorphic maps. More precisely, if $\tau : X_1 \to X_2$ is a holomorphic map between complex manifolds and φ is a form on X_2 then $df^*(\varphi) = f^*(d\varphi)$, $dd^c f^*(\varphi) = f^*(dd^c\varphi)$, etc. Recall that the form φ is *closed* (resp. ∂-closed, $\overline{\partial}$-closed, dd^c-closed) if $d\varphi$ (resp. $\partial\varphi$, $\overline{\partial}\varphi$, $dd^c\varphi$) vanishes. The form φ is *exact* (resp. ∂-exact, $\overline{\partial}$-exact, dd^c-exact) if it is equal to the differential $d\psi$ (resp. $\partial\psi$, $\overline{\partial}\psi$, $dd^c\psi$) of a form ψ. Clearly, exact forms are closed.

A smooth $(1,1)$-form ω on X is *Hermitian* if it can be written in local coordinates as

$$\omega(z) = \sqrt{-1} \sum_{1 \le i,j \le k} \alpha_{ij}(z) dz_i \wedge d\overline{z}_j,$$

where α_{ij} are smooth functions such that the matrix (α_{ij}) is Hermitian. We consider a form ω such that the matrix (α_{ij}) is positive definite at every point. It is strictly positive in the sense that we will introduce later. If a is a point in X, we can find local coordinates z such that $z = 0$ at a and ω is equal near 0 to the Euclidean form $dd^c\|z\|^2$ modulo a term of order $\|z\|$. The form ω is always real and induces a norm on the tangent spaces of X. So, it defines a Riemannian metric on X. We say that ω is *a Kähler form* if it is a closed positive definite Hermitian form. In this case, one can find local coordinates z such that $z = 0$ at a and ω is equal near 0 to $dd^c\|z\|^2$ modulo a term of order $\|z\|^2$. So, at the infinitesimal level, a Kähler metric is close to the Euclidean one. This is a crucial property in Hodge theory in the complex setting.

Consider now a compact complex manifold X of dimension k. Assume that X is *a Kähler manifold*, i.e. it admits a Kähler form ω. Recall that *the de Rham cohomology group $H^l(X,\mathbb{C})$ is the quotient of the space of closed l-forms by the subspace of exact l-forms*. This complex vector space is of finite dimension. The real groups $H^l(X,\mathbb{R})$ are defined in the same way using real forms. We have

$$H^l(X,\mathbb{C}) = H^l(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

If α is a closed l-form, its class in $H^l(X,\mathbb{C})$ is denoted by $[\alpha]$. The group $H^0(X,\mathbb{C})$ is just the set of constant functions. So, it is isomorphic to $\mathbb{C}$. The group $H^{2k}(X,\mathbb{C})$ is also isomorphic to $\mathbb{C}$. The isomorphism is given by the canonical map $[\alpha] \mapsto \int_X \alpha$. For l,m such that $l + m \le 2k$, *the cup-product*

$$\smile : H^l(X,\mathbb{C}) \times H^m(X,\mathbb{C}) \to H^{l+m}(X,\mathbb{C})$$

is defined by $[\alpha] \smile [\beta] := [\alpha \wedge \beta]$. The Poincaré duality theorem says that the cup-product is a non-degenerated bilinear form when $l + m = 2k$. So, it defines an isomorphism between $H^l(X,\mathbb{C})$ and the dual of $H^{2k-l}(X,\mathbb{C})$.

Let $H^{p,q}(X,\mathbb{C})$, $0 \leq p,q \leq k$, denote the subspace of $H^{p+q}(X,\mathbb{C})$ generated by the classes of closed (p,q)-forms. We call $H^{p,q}(X,\mathbb{C})$ the *Hodge cohomology group*. Hodge theory shows that

$$H^l(X,\mathbb{C}) = \bigoplus_{p+q=l} H^{p,q}(X,\mathbb{C}) \qquad \text{and} \qquad H^{q,p}(X,\mathbb{C}) = \overline{H^{p,q}(X,\mathbb{C})}.$$

This, together with the Poincaré duality, induces a canonical isomorphism between $H^{p,q}(X,\mathbb{C})$ and the dual space of $H^{k-p,k-q}(X,\mathbb{C})$. Define for $p = q$

$$H^{p,p}(X,\mathbb{R}) := H^{p,p}(X,\mathbb{C}) \cap H^{2p}(X,\mathbb{R}).$$

We have

$$H^{p,p}(X,\mathbb{C}) = H^{p,p}(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

Recall that *the Dolbeault cohomology group $H_{\bar{\partial}}^{p,q}(X)$* is the quotient of the space of $\bar{\partial}$-closed (p,q)-forms by the subspace of $\bar{\partial}$-exact (p,q)-forms. Observe that a (p,q)-form is d-closed if and only if it is ∂-closed and $\bar{\partial}$-closed. Therefore, there is a natural morphism between the Hodge and the Dolbeault cohomology groups. Hodge theory asserts that this is in fact an isomorphism: we have

$$H^{p,q}(X,\mathbb{C}) \simeq H_{\bar{\partial}}^{p,q}(X).$$

The result is a consequence of the following theorem, the so-called dd^c-*lemma*, see e.g. [DEM, VO].

Theorem A.1. *Let φ be a smooth d-closed (p,q)-form on X. Then φ is dd^c-exact if and only if it is d-exact (or ∂-exact or $\bar{\partial}$-exact).*

The projective space $\mathbb{P}^k$ admits a Kähler form ω_{FS}, called *the Fubini-Study form*. It is defined on the chart U_i by

$$\omega_{\text{FS}} := dd^c \log \left(\sum_{j=0}^{k} \left| \frac{z_j}{z_i} \right|^2 \right).$$

In other words, if $\pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k$ is the canonical projection, then ω_{FS} is defined by

$$\pi^*(\omega_{\text{FS}}) := dd^c \log \left(\sum_{i=0}^{k} |z_i|^2 \right).$$

One can check that ω_{FS}^k is a probability measure on $\mathbb{P}^k$. The cohomology groups of $\mathbb{P}^k$ are very simple. We have $H^{p,q}(\mathbb{P}^k,\mathbb{C}) = 0$ for $p \neq q$ and $H^{p,p}(\mathbb{P}^k,\mathbb{C}) \simeq \mathbb{C}$. The groups $H^{p,p}(\mathbb{P}^k,\mathbb{R})$ and $H^{p,p}(\mathbb{P}^k,\mathbb{C})$ are generated by the class of ω_{FS}^p. Submanifolds of $\mathbb{P}^k$ are Kähler, as submanifolds of a Kähler manifold. Chow's theorem says that such a manifold is algebraic, i.e. it is the set of common zeros of a finite family of homogeneous polynomials in z. A compact manifold is *projective* if it is bi-holomorphic to a submanifold of a projective space. Their cohomology groups are in general very rich and difficult to describe.

A useful result of Blanchard [BN] says that the blow-up of a compact Kähler manifold along a submanifold is always a compact Kähler manifold. The construction of the blow-up is as follows. Consider first the case of open sets in $\mathbb{C}^k$ with $k \geq 2$. Observe that $\mathbb{C}^k$ is the union of the complex lines passing through 0 which are parametrized by the projective space $\mathbb{P}^{k-1}$. *The blow-up $\widehat{\mathbb{C}^k}$ of $\mathbb{C}^k$ at 0 is* obtained by separating these complex lines, that is, we keep $\mathbb{C}^k \setminus \{0\}$ and replace 0 with a copy of $\mathbb{P}^{k-1}$. More precisely, if $z = (z_1, \ldots, z_k)$ denote the coordinates of $\mathbb{C}^k$ and $[w] = [w_1 : \cdots : w_k]$ are homogeneous coordinates of $\mathbb{P}^{k-1}$, then $\widehat{\mathbb{C}^k}$ is the submanifold of $\mathbb{C}^k \times \mathbb{P}^{k-1}$ defined by the equations $z_i w_j = z_j w_i$ for $1 \leq i, j \leq k$. If U is an open set in $\mathbb{C}^k$ containing 0, the blow-up $\widehat{U}$ of U at 0 is defined by $\pi^{-1}(U)$ where $\pi : \widehat{\mathbb{C}^k} \to \mathbb{C}^k$ is the canonical projection.

If U is a neighbourhood of 0 in $\mathbb{C}^{k-p}$, $p \leq k-2$, and V is an open set in $\mathbb{C}^p$, then the blow-up of $U \times V$ along $\{0\} \times V$ is equal to $\widehat{U} \times V$. Consider now a submanifold Y of X of dimension $p \leq k-2$. We cover X by charts which either do not intersect Y or are of the type $U \times V$, where Y is identified with $\{0\} \times V$. *The blow-up $\widehat{X}$ is* obtained by sticking the charts outside Y with the blow-ups of charts which intersect Y. The natural projection $\pi : \widehat{X} \to X$ defines a bi-holomorphic map between $\widehat{X} \setminus \pi^{-1}(Y)$ and $X \setminus Y$. The set $\pi^{-1}(Y)$ is a smooth hypersurface, i.e. submanifold of codimension 1; it is called *the exceptional hypersurface*. Blow-up may be defined using the local ideals of holomorphic functions vanishing on Y. The blow-up of a projective manifold along a submanifold is a projective manifold.

We now recall some facts on analytic sets, see [GU, N]. Let X be an arbitrary complex manifold of dimension k[9]. Analytic sets of X can be seen as submanifolds of X, possibly with singularities. Analytic sets of dimension 0 are locally finite subsets, those of dimension 1 are (possibly singular) Riemann surfaces. For example, $\{z_1^2 = z_2^3\}$ is an analytic set of $\mathbb{C}^2$ of dimension 1 with a singularity at 0. Chow's theorem holds for analytic sets: any analytic set in $\mathbb{P}^k$ is the set of common zeros of a finite family of homogeneous polynomials.

Recall that an *analytic set* Y of X is locally the set of common zeros of holomorphic functions: for every point $a \in X$ there is a neighbourhood U of a and holomorphic functions f_i on U such that $Y \cap U$ is the intersection of $\{f_i = 0\}$. We can choose U so that $Y \cap U$ is defined by a finite family of holomorphic functions. Analytic sets are closed for the usual topology on X. Local rings of holomorphic functions on X induce local rings of holomorphic functions on Y. An analytic set Y is *irreducible* if it is not a union of two different non-empty analytic sets of X. A general analytic set Y can be decomposed in a unique way into a union of irreducible analytic subsets $Y = \cup Y_i$, where no component Y_i is contained in another one. The decomposition is locally finite, that is, given a compact set K in X, only finitely many Y_i intersect K.

Any increasing sequence of irreducible analytic subsets of X is stationary. A decreasing sequence (Y_n) of analytic subsets of X is always locally stationary, that is, for any compact subset K of X, the sequence $(Y_n \cap K)$ is stationary. Here, we do not suppose Y_n irreducible. The topology on X whose closed sets are exactly the analytic sets, is called *the Zariski topology*. When X is connected, non-empty open

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[9] We often assume implicitly that X is connected for simplicity.

Zariski sets are dense in X for the usual topology. The restriction of the Zariski topology on X to Y is also called the *Zariski topology* of Y. When Y is irreducible, the non-empty Zariski open subsets are also dense in Y but this is not the case for reducible analytic sets.

There is a minimal analytic subset $\text{sing}(Y)$ in X such that $Y \setminus \text{sing}(Y)$ is a (smooth) complex submanifold of $X \setminus \text{sing}(Y)$, i.e. a complex manifold which is closed and without boundary in $X \setminus \text{sing}(Y)$. The analytic set $\text{sing}(Y)$ is the *singular part* of Y. The *regular part* of Y is denoted by $\text{reg}(Y)$; it is equal to $Y \setminus \text{sing}(Y)$. The manifold $\text{reg}(Y)$ is not necessarily irreducible; it may have several components. We call *dimension of Y*, $\dim(Y)$, the maximum of the dimensions of these components; the *codimension* $\text{codim}(Y)$ of Y in X is the integer $k - \dim(Y)$. We say that Y is a proper analytic set of X if it has positive codimension. When all the components of Y have the same dimension, we say that Y is *of pure dimension* or *of pure codimension*. When $\text{sing}(Y)$ is non-empty, its dimension is always strictly smaller than the dimension of Y. We can again decompose $\text{sing}(Y)$ into regular and singular parts. The procedure can be repeated less than k times and gives a stratification of Y into disjoint complex manifolds. Note that Y is irreducible if and only if $\text{reg}(Y)$ is a connected manifold. The following result is due to Wirtinger.

Theorem A.2 (Wirtinger). *Let Y be analytic set of pure dimension p of a Hermitian manifold (X, ω). Then the $2p$-dimensional volume of Y on a Borel set K is equal to*

$$\text{volume}(Y \cap K) = \frac{1}{p!} \int_{\text{reg}(Y) \cap K} \omega^p.$$

Here, the volume is with respect to the Riemannian metric induced by ω.

Let D_k denote the unit polydisc $\{|z_1| < 1, \ldots, |z_k| < 1\}$ in $\mathbb{C}^k$. The following result describes the local structure of analytic sets.

Theorem A.3. *Let Y be an analytic set of pure dimension p of X. Let a be a point of Y. Then there is a holomorphic chart U of X, bi-holomorphic to D_k, with local coordinates $z = (z_1, \ldots, z_k)$, such that $z = 0$ at a, U is given by $\{|z_1| < 1, \ldots, |z_k| < 1\}$ and the projection $\pi : U \to D_p$, defined by $\pi(z) := (z_1, \ldots, z_p)$, is proper on $Y \cap U$. In this case, there is a proper analytic subset S of D_p such that $\pi : Y \cap U \setminus \pi^{-1}(S) \to D_p \setminus S$ is a finite covering and the singularities of Y are contained in $\pi^{-1}(S)$.*

Recall that a holomorphic map $\tau : X_1 \to X_2$ between complex manifolds of the same dimension is *a covering of degree d* if each point of X_2 admits a neighbourhood V such that $\tau^{-1}(V)$ is a disjoint union of d open sets, each of which is sent bi-holomorphically to V by τ. Observe the previous theorem also implies that the fibers of $\pi : Y \cap U \to D_p$ are finite and contain at most d points if d is the degree of the covering. We can reduce U in order to have that a is the unique point in the fiber $\pi^{-1}(0) \cap Y$. The degree d of the covering depends on the choice of coordinates and the smallest integer d obtained in this way is called *the multiplicity* of Y at a and is denoted by $\text{mult}(Y, a)$. We will see that $\text{mult}(Y, a)$ is the Lelong number at a of the positive closed current associated to Y. In other words, if B_r denotes the ball of

center a and of radius r, then the ratio between the volume of $Y \cap B_r$ and the volume of a ball of radius r in $\mathbb{C}^p$ decreases to $\mathrm{mult}(Y, a)$ when r decreases to 0.

Let $\tau : X_1 \to X_2$ be an open holomorphic map between complex manifolds of the same dimension. Applying the above result to the graph of τ, we can show that for any point $a \in X_1$ and for a neighbourhood U of a small enough, if z is a generic point in X_2 close enough to $\tau(a)$, the number of points in $\tau^{-1}(z) \cap U$ does not depend on z. We call this number *the multiplicity* or *the local topological degree* of τ at a. We say that τ is *a ramified covering of degree d* if τ is open, proper and each fiber of τ contains exactly d points counted with multiplicity. In this case, if Σ_2 is the set of critical values of τ and $\Sigma_1 := \tau^{-1}(\Sigma_2)$, then $\tau : X_1 \setminus \Sigma_1 \to X_2 \setminus \Sigma_2$ is a covering of degree d.

We recall the notion of analytic space which generalizes complex manifolds and their analytic subsets. An *analytic space of dimension* $\leq p$ is defined as a complex manifold but a chart is replaced with an analytic subset of dimension $\leq p$ in an open set of a complex Euclidean space. As in the case of analytic subsets, one can decompose analytic spaces into irreducible components and into regular and singular parts. The notions of dimension, of Zariski topology and of holomorphic maps can be extended to analytic spaces. The precise definition uses the local ring of holomorphic functions, see [GU, N]. An analytic space is *normal* if the local ring of holomorphic functions at every point is integrally closed. This is equivalent to the fact that for U open in Z holomorphic functions on $\mathrm{reg}(Z) \cap U$ which are bounded near $\mathrm{sing}(Z) \cap U$, are holomorphic on U. In particular, normal analytic spaces are locally irreducible. A holomorphic map $f : Z_1 \to Z_2$ between complex spaces is a continuous map which induces morphisms from local rings of holomorphic functions on Z_2 to the ones on Z_1. The notions of ramified covering, of multiplicity and of open maps can be extended to normal analytic spaces. We have the following useful result where $\widetilde{Z}$ is called *normalization* of Z.

Theorem A.4. *Let Z be an analytic space. Then there is a unique, up to a bi-holomorphic map, normal analytic space $\widetilde{Z}$ and a finite holomorphic map $\pi : \widetilde{Z} \to Z$ such that*

1. *$\pi^{-1}(\mathrm{reg}(Z))$ is a dense Zariski open set of $\widetilde{Z}$ and π defines a bi-holomorphic map between $\pi^{-1}(\mathrm{reg}(Z))$ and $\mathrm{reg}(Z)$;*
2. *If $\tau : Z' \to Z$ is a holomorphic map between analytic spaces, then there is a unique holomorphic map $h : Z' \to \widetilde{Z}$ satisfying $\pi \circ h = \tau$.*

In particular, holomorphic self-maps of Z can be lifted to holomorphic self-maps of $\widetilde{Z}$.

Example A.5. Let $\pi : \mathbb{C} \to \mathbb{C}^2$ be the holomorphic map given by $\pi(t) = (t^2, t^3)$. This map defines a normalization of the analytic curve $\{z_1^3 = z_2^2\}$ in $\mathbb{C}^2$ which is singular at 0. The normalization of the analytic set $\{z_1 = 0\} \cup \{z_1^3 = z_2^2\}$ is the union of two disjoint complex lines. The normalization of a complex curve (an analytic set of pure dimension 1) is always smooth.

The following desingularization theorem, due to Hironaka, is very useful.

Theorem A.6. *Let Z be an analytic space. Then there is a smooth manifold $\widehat{Z}$, possibly reducible, and a holomorphic map $\pi : \widehat{Z} \to Z$ such that $\pi^{-1}(\mathrm{reg}(Z))$ is a dense Zariski open set of $\widehat{Z}$ and π defines a bi-holomorphic map between $\pi^{-1}(\mathrm{reg}(Z))$ and $\mathrm{reg}(Z)$.*

When Z is an analytic subset of a manifold X, then one can obtain a map $\pi : \widehat{X} \to X$ using a sequence of blow-ups along the singularities of Z. The manifold $\widehat{Z}$ is the strict transform of Z by π. The difference with the normalization of Z is that we do not have the second property in Theorem A.4 but $\widehat{Z}$ is smooth.

Exercise A.7. Let X be a compact Kähler manifold of dimension k. Show that the Betti number b_l, i.e. the dimension of $H^l(X, \mathbb{R})$, is even if l is odd and does not vanish if l is even.

Exercise A.8. Let $\mathrm{Grass}(l, k)$ denote the Grassmannian, i.e. the set of linear subspaces of dimension l of $\mathbb{C}^k$. Show that $\mathrm{Grass}(l, k)$ admits a natural structure of a projective manifold.

Exercise A.9. Let X be a compact complex manifold of dimension ≥ 2 and $\pi : \widehat{X \times X} \to X \times X$ the blow-up of $X \times X$ along the diagonal Δ. Let Π_1, Π_2 denote the natural projections from $\widehat{X \times X}$ onto the two factors X of $X \times X$. Show that Π_1, Π_2 and their restrictions to $\pi^{-1}(\Delta)$ are submersions.

Exercise A.10. Let E be a finite or countable union of proper analytic subsets of a connected manifold X. Show that $X \setminus E$ is connected and dense in X for the usual topology.

Exercise A.11. Let $\tau : X_1 \to X_2$ be a ramified covering of degree n. Let φ be a function on X_1. Define

$$\tau_*(\varphi)(z) := \sum_{w \in \tau^{-1}(z)} \varphi(w),$$

where the points in $\tau^{-1}(z)$ are counted with multiplicity. If φ is upper semi-continuous or continuous, show that $\tau_*(\varphi)$ is upper semi-continuous or continuous respectively. Show that the result still holds for a general open map τ between manifolds of the same dimension if φ has compact support in X_1.

A.2 Positive Currents and p.s.h. Functions

In this paragraph, we introduce positive forms, positive currents and plurisubharmonic functions on complex manifolds. The concept of positivity and the notion of plurisubharmonic functions are due to Lelong and Oka. The theory has many applications in complex algebraic geometry and in dynamics.

Let X be a complex manifold of dimension k and ω a Hermitian $(1,1)$-form on X which is positive definite at every point. Recall that a current S on X, of degree l and of dimension $2k-l$, is a continuous linear form on the space $\mathscr{D}^{2k-l}(X)$ of smooth $(2k-l)$-forms with compact support in X. Its value on a $(2k-l)$-form $\varphi \in \mathscr{D}^{2k-l}(X)$ is denoted by $S(\varphi)$ or more frequently by $\langle S, \varphi \rangle$. On a chart, S corresponds to a continuous linear form acting on the coefficients of φ. So, it can be represented as an l-form with distribution coefficients. A sequence (S_n) of l-currents converges to an l-current S if for every $\varphi \in \mathscr{D}^{2k-l}(X)$, $\langle S_n, \varphi \rangle$ converge to $\langle S, \varphi \rangle$. The conjugate of S is the l-current $\overline{S}$ defined by

$$\langle \overline{S}, \varphi \rangle := \overline{\langle S, \overline{\varphi} \rangle},$$

for $\varphi \in \mathscr{D}^{2k-l}(X)$. The current S is real if and only if $\overline{S} = S$.

The support of S is the smallest closed subset $\mathrm{supp}(S)$ of X such that $\langle S, \varphi \rangle = 0$ when φ is supported on $X \setminus \mathrm{supp}(S)$. The current S extends continuously to the space of smooth forms φ such that $\mathrm{supp}(\varphi) \cap \mathrm{supp}(S)$ is compact in X. If X' is a complex manifold of dimension k' with $2k' \geq 2k-l$, and if $\tau : X \to X'$ is a holomorphic map which is proper on the support of S, we can define *the push-forward* $\tau_*(S)$ of S by τ. This is a current $\tau_*(S)$ of the same dimension than S, i.e. of degree $2k'-2k+l$, which is supported on $\tau(\mathrm{supp}(S))$, it satisfies

$$\langle \tau_*(S), \varphi \rangle := \langle S, \tau^*(\varphi) \rangle$$

for $\varphi \in \mathscr{D}^{2k-l}(X')$. If X' is a complex manifold of dimension $k' \geq k$ and if $\tau : X' \to X$ is a submersion, we can define *the pull-back* $\tau^*(S)$ of S by τ. This is an l-current supported on $\tau^{-1}(\mathrm{supp}(S))$, it satisfies

$$\langle \tau^*(S), \varphi \rangle := \langle S, \tau_*(\varphi) \rangle$$

for $\varphi \in \mathscr{D}^{2k'-l}(X')$. Indeed, since τ is a submersion, the current $\tau_*(\varphi)$ is in fact a smooth form with compact support in X; it is given by an integral of φ on the fibers of τ.

Any smooth differential l-form ψ on X defines a current: it defines the continuous linear form $\varphi \mapsto \int_X \psi \wedge \varphi$ on $\varphi \in \mathscr{D}^{2k-l}(X)$. So, currents extend the notion of differential forms. The operators $d, \partial, \overline{\partial}$ on differential forms extend to currents. For example, we have that dS is an $(l+1)$-current defined by

$$\langle dS, \varphi \rangle := (-1)^{l+1} \langle S, d\varphi \rangle$$

for $\varphi \in \mathscr{D}^{2k-l-1}(X)$. One easily check that when S is a smooth form, the above identity is a consequence of the Stokes' formula. We say that S is of *bidegree* (p,q) and *of bidimension* $(k-p, k-q)$ if it vanishes on forms of bidegree $(r,s) \neq (k-p, k-q)$. The conjugate of a (p,q)-current is of bidegree (q,p). So, if such a current is real, we have necessarily $p = q$. Note that the push-forward and the pull-back by holomorphic maps commute with the above operators. They preserve real currents; the push-forward preserves the bidimension and the pull-back preserves the bidegree.

There are three notions of positivity which coincide for the bidegrees $(0,0)$, $(1,1)$, $(k-1,k-1)$ and (k,k). Here, we only use two of them. They are dual to each other. A (p,p)-form φ is *(strongly) positive* if at each point, it is equal to a combination with positive coefficients of forms of type

$$(\sqrt{-1}\alpha_1 \wedge \overline{\alpha}_1) \wedge \ldots \wedge (\sqrt{-1}\alpha_p \wedge \overline{\alpha}_p),$$

where α_i are $(1,0)$-forms. Any (p,p)-form can be written as a finite combination of positive (p,p)-forms. For example, in local coordinates z, a $(1,1)$-form ω is written as

$$\omega = \sum_{i,j=1}^{k} \alpha_{ij}\sqrt{-1}dz_i \wedge d\overline{z}_j,$$

where α_{ij} are functions. This form is positive if and only if the matrix (α_{ij}) is positive semi-definite at every point. In local coordinates z, the $(1,1)$-form $dd^c\|z\|^2$ is positive. One can write $dz_1 \wedge d\overline{z}_2$ as a combination of $dz_1 \wedge d\overline{z}_1$, $dz_2 \wedge d\overline{z}_2$, $d(z_1 \pm z_2) \wedge \overline{d(z_1 \pm z_2)}$ and $d(z_1 \pm \sqrt{-1}z_2) \wedge \overline{d(z_1 \pm \sqrt{-1}z_2)}$. Hence, positive forms generate the space of (p,p)-forms.

A (p,p)-current S is *weakly positive* if for every smooth positive $(k-p,k-p)$-form φ, $S \wedge \varphi$ is a positive measure and is *positive* if $S \wedge \varphi$ is a positive measure for every smooth weakly positive $(k-p,k-p)$-form φ. Positivity implies weak positivity. These properties are preserved under pull-back by holomorphic submersions and push-forward by proper holomorphic maps. Positive and weakly positive forms or currents are real. One can consider positive and weakly positive (p,p)-forms as sections of some bundles of salient convex closed cones which are contained in the real part of the vector bundle $\bigwedge^p \Omega^{1,0} \otimes \bigwedge^p \Omega^{0,1}$.

The wedge-product of a positive current with a positive form is positive. The wedge-product of a weakly positive current with a positive form is weakly positive. Wedge-products of weakly positive forms or currents are not always weakly positive. For real (p,p)-currents or forms S, S', we will write $S \geq S'$ and $S' \leq S$ if $S - S'$ is positive. A current S is *negative* if $-S$ is positive. A (p,p)-current or form S is *strictly positive* if in local coordinates z, there is a constant $\varepsilon > 0$ such that $S \geq \varepsilon(dd^c\|z\|^2)^p$. Equivalently, S is strictly positive if we have locally $S \geq \varepsilon\omega^p$ with $\varepsilon > 0$.

Example A.12. Let Y be an analytic set of pure codimension p of X. Using the local description of Y near a singularity in Theorem A.3 and Wirtinger's theorem A.2, one can prove that the $2(k-p)$-dimensional volume of Y is locally finite in X. This allows to define the following (p,p)-current $[Y]$ by

$$\langle [Y], \varphi \rangle := \int_{\mathrm{reg}(Y)} \varphi$$

for φ in $\mathscr{D}^{k-p,k-p}(X)$, the space of smooth $(k-p,k-p)$-forms with compact support in X. Lelong proved that this current is positive and closed [DEM, LE].

If S is a (weakly) positive (p,p)-current, it is of order 0, i.e. it extends continuously to the space of continuous forms with compact support in X. In other words, on a chart of X, the current S corresponds to a differential form with measure coefficients. We define the *mass* of S on a Borel set K by

$$\|S\|_K := \int_K S \wedge \omega^{k-p}.$$

When K is relatively compact in X, we obtain an equivalent norm if we change the Hermitian metric on X. This is a consequence of the property we mentioned above, which says that S takes values in salient convex closed cones. Note that the previous mass-norm is just defined by an integral, which is easier to compute or to estimate than the usual mass for currents on real manifolds.

Positivity implies an important compactness property. As for positive measures, any family of positive (p,p)-currents with locally uniformly bounded mass, is relatively compact in the cone of positive (p,p)-currents. For the current $[Y]$ in Example A.12, by Wirtinger's theorem, the mass on K is equal to $(k-p)!$ times the volume of $Y \cap K$ with respect to the considered Hermitian metric. If S is a negative (p,p)-current, its mass is defined by

$$\|S\|_K := -\int_K S \wedge \omega^{k-p}.$$

The following result is the complex version of the classical support theorem in the real setting, [BA, HP, FE].

Proposition A.13. *Let S be a (p,p)-current supported on a smooth complex submanifold Y of X. Let $\tau : Y \to X$ denote the inclusion map. Assume that S is $\mathbb{C}$-normal, i.e. S and $dd^c S$ are of order 0. Then, S is a current on Y. More precisely, there is a $\mathbb{C}$-normal (p,p)-current S' on Y such that $S = \tau_*(S')$. If S is positive closed and Y is of dimension $k - p$, then S is equal to a combination with positive coefficients of currents of integration on components of Y.*

The last property holds also when Y is a singular analytic set. Proposition A.13 applies to positive closed (p,p)-currents which play an important role in complex geometry and dynamics. These currents generalize analytic sets of dimension $k - p$, as we have seen in Example A.12. They have no mass on Borel sets of $2(k-p)$-dimensional Hausdorff measure 0. The proposition is used in order to develop a calculus on potentials of closed currents.

We introduce now the notion of Lelong number for such currents which generalizes the notion of multiplicity for analytic sets. Let S be a positive closed (p,p)-current on X. Consider local coordinates z on a chart U of X and the local Kähler form $dd^c \|z\|^2$. Let $B_a(r)$ denote the ball of center a and of radius r contained in U. Then, $S \wedge (dd^c \|z\|^2)^{k-p}$ is a positive measure on U. Define for $a \in U$

$$\nu(S,a,r) := \frac{\|S \wedge (dd^c \|z\|^2)^{k-p}\|_{B_a(r)}}{\pi^{k-p} r^{2(k-p)}}.$$

Note that $\pi^{k-p}r^{2(k-p)}$ is $(k-p)!$ times the volume of a ball in $\mathbb{C}^{k-p}$ of radius r, i.e. the mass of the current associated to this ball. When r decreases to 0, $v(S,a,r)$ is decreasing and the *Lelong number* of S at a is the limit

$$v(S,a) := \lim_{r \to 0} v(S,a,r).$$

It does not depend on the coordinates. So, we can define the Lelong number for currents on any manifold. Note that $v(S,a)$ is also the mass of the measure $S \wedge (dd^c \log \|z-a\|)^{k-p}$ at a. We will discuss the wedge-product (intersection) of currents in the next paragraph.

If S is the current of integration on an analytic set Y, by Thie's theorem, $v(S,a)$ is equal to the multiplicity of Y at a which is an integer. This implies the following Lelong's inequality: *the Euclidean $2(k-p)$-dimensional volume of Y in a ball $B_a(r)$ centered at a point $a \in Y$, is at least equal to* $\frac{1}{(k-p)!}\pi^{k-p}r^{2(k-p)}$, *the volume in $B_a(r)$ of a $(k-p)$-dimensional linear space passing through a.*

Positive closed currents generalize analytic sets but they are much more flexible. A remarkable fact is that the use of positive closed currents allows to construct analytic sets. The following theorem of Siu [SIU] is a beautiful application of the complex L^2 method.

Theorem A.14. *Let S be a positive closed (p,p)-current on X. Then, for $c > 0$, the level set $\{v(S,a) \geq c\}$ of the Lelong number is an analytic set of X, of dimension $\leq k-p$. Moreover, there is a unique decomposition $S = S_1 + S_2$ where S_1 is a locally finite combination, with positive coefficients, of currents of integration on analytic sets of codimension p and S_2 is a positive closed (p,p)-current such that $\{v(S_2,z) > 0\}$ is a finite or countable union of analytic sets of dimension $\leq k-p-1$.*

Calculus on currents is often delicate. However, the theory is well developped for positive closed $(1,1)$-currents thanks to the use of plurisubharmonic functions. Note that positive closed $(1,1)$-currents correspond to hypersurfaces (analytic sets of pure codimension 1) in complex geometry and working with (p,p)-currents, as with higher codimension analytic sets, is more difficult.

An upper semi-continuous function $u : X \to \mathbb{R} \cup \{-\infty\}$, not identically $-\infty$ on any component of X, is *plurisubharmonic* (p.s.h. for short) if it is subharmonic or identically $-\infty$ on any holomorphic disc in X. Recall that a *holomorphic disc* in X is a holomorphic map $\tau : \Delta \to X$ where Δ is the unit disc in $\mathbb{C}$. One often identifies this holomorphic disc with its image $\tau(\Delta)$. If u is p.s.h., then $u \circ \tau$ is subharmonic or identically $-\infty$ on Δ. As for subharmonic functions, we have the submean inequality: *in local coordinates, the value at a of a p.s.h. function is smaller or equal to the average of the function on a sphere centered at a.* Indeed, this average increases with the radius of the sphere. The submean inequality implies the maximum principle: *if a p.s.h. function on a connected manifold X has a maximum, it is constant.* The semi-continuity implies that p.s.h. functions are locally bounded from above. A function v is *pluriharmonic* if v and $-v$ are p.s.h. Pluriharmonic functions are locally real parts of holomorphic functions, in particular, they are real analytic. The following proposition is of constant use.

Proposition A.15. *A function $u : X \to \mathbb{R} \cup \{-\infty\}$ is p.s.h. if and only if the following conditions are satisfied*

1. *u is strongly upper semi-continuous, that is, for any subset A of full Lebesgue measure in X and for any point a in X, we have $u(a) = \limsup u(z)$ when $z \to a$ and $z \in A$.*
2. *u is locally integrable with respect to the Lebesgue measure on X and $dd^c u$ is a positive closed $(1,1)$-current.*

Conversely, any positive closed $(1,1)$-current can be locally written as $dd^c u$ where u is a (local) p.s.h. function. This function is called *a local potential* of the current. Two local potentials differ by a pluriharmonic function. So, there is almost a correspondence between positive closed $(1,1)$-currents and p.s.h. functions. We say that u is *strictly p.s.h.* if $dd^c u$ is strictly positive. The p.s.h. functions are defined at every point; this is a crucial property in pluripotential theory. Other important properties of this class of functions are some strong compactness properties that we state below.

If S is a positive closed (p, p)-current, one can write locally $S = dd^c U$ with U a $(p-1, p-1)$-current. We can choose the potential U negative with good estimates on the mass but the difference of two potentials may be very singular. The use of potentials U is much more delicate than in the bidegree $(1,1)$ case. We state here a useful local estimate, see e.g. [DN].

Proposition A.16. *Let V be convex open domain in $\mathbb{C}^k$ and W an open set with $W \Subset V$. Let S be a positive closed (p, p)-current on V. Then there is a negative L^1 form U of bidegree $(p-1, p-1)$ on W such that $dd^c U = S$ and $\|U\|_{L^1(W)} \leq c\|S\|_V$ where $c > 0$ is a constant independent of S. Moreover, U depends continuously on S, where the continuity is with respect to the weak topology on S and the $L^1(W)$ topology on U.*

Note that when $p = 1$, U is equal almost everywhere to a p.s.h. function u such that $dd^c u = S$.

Example A.17. Let f be a holomorphic function on X not identically 0 on any component of X. Then, $\log|f|$ is a p.s.h. function and we have $dd^c \log|f| = \sum n_i[Z_i]$ where Z_i are irreducible components of the hypersurface $\{f = 0\}$ and n_i their multiplicities. The last equation is called *Poincaré-Lelong equation*. Locally, the ideal of holomorphic functions vanishing on Z_i is generated by a holomorphic function g_i and f is equal to the product of $\prod g_i^{n_i}$ with a non-vanishing holomorphic function. In some sense, $\log|f|$ is one of the most singular p.s.h. functions. If X is a ball, the convex set generated by such functions is dense in the cone of p.s.h. functions [HO, GU] for the L^1_{loc} topology. If $f_1, \ldots, f_n$ are holomorphic on X, not identically 0 on a component of X, then $\log(|f_1|^2 + \cdots + |f_n|^2)$ is also a p.s.h. function.

The following proposition is useful in constructing p.s.h. functions.

Proposition A.18. *Let χ be a function defined on $(\mathbb{R} \cup \{-\infty\})^n$ with values in $\mathbb{R} \cup \{-\infty\}$, not identically $-\infty$, which is convex in all variables and increasing in*

each variable. Let $u_1, \ldots, u_n$ be p.s.h. functions on X. Then $\chi(u_1, \ldots, u_n)$ is p.s.h. In particular, the function $\max(u_1, \ldots, u_n)$ is p.s.h.

We call *complete pluripolar set* the pole set $\{u = -\infty\}$ of a p.s.h. function and *pluripolar set* a subset of a complete pluripolar one. Pluripolar sets are of Hausdorff dimension $\leq 2k - 2$, in particular, they have zero Lebesgue measure. Finite and countable unions of (locally) pluripolar sets are (locally) pluripolar. In particular, finite and countable unions of analytic subsets are locally pluripolar.

Proposition A.19. *Let E be a closed pluripolar set in X and u a p.s.h. function on $X \setminus E$, locally bounded above near E. Then the extension of u to X given by*

$$u(z) := \limsup_{\substack{w \to z \\ w \in X \setminus E}} u(w) \quad for \quad z \in E,$$

is a p.s.h. function.

The following result describes compactness properties of p.s.h. functions, see [HO].

Proposition A.20. *Let (u_n) be a sequence of p.s.h. functions on X, locally bounded from above. Then either it converges locally uniformly to $-\infty$ on a component of X or there is a subsequence (u_{n_i}) which converges in $L^p_{loc}(X)$ to a p.s.h. function u for every p with $1 \leq p < \infty$. In the second case, we have $\limsup u_{n_i} \leq u$ with equality outside a pluripolar set. Moreover, if K is a compact subset of X and if h is a continuous function on K such that $u < h$ on K, then $u_{n_i} < h$ on K for i large enough.*

The last assertion is the classical Hartogs' lemma. It suggests the following notion of convergence introduced in [DS10]. Let (u_n) be a sequence of p.s.h. functions converging to a p.s.h. function u in $L^1_{loc}(X)$. We say that the sequence (u_n) *converges in the Hartogs' sense* or *is H-convergent* if for any compact subset K of X there are constants c_n converging to 0 such that $u_n + c_n \geq u$ on K. In this case, Hartogs' lemma implies that u_n converge pointwise to u. If (u_n) decreases to a function u, not identically $-\infty$, then u is p.s.h. and (u_n) converges in the Hartogs' sense. The following result is useful in the calculus with p.s.h. functions.

Proposition A.21. *Let u be a p.s.h. function on an open subset D of $\mathbb{C}^k$. Let $D' \Subset D$ be an open set. Then, there is a sequence of smooth p.s.h. functions u_n on D' which decreases to u.*

The functions u_n can be obtained as the standard convolution of u with some radial function ρ_n on $\mathbb{C}^k$. The submean inequality for u allows to choose ρ_n so that u_n decrease to u.

The following result, see [HO2], may be considered as the strongest compactness property for p.s.h. functions. The proof can be reduced to the one dimensional case by slicing.

Theorem A.22. *Let $\mathscr{F}$ be a family of p.s.h. functions on X which is bounded in $L^1_{loc}(X)$. Let K be a compact subset of X. Then there are constants $\alpha > 0$ and $A > 0$ such that*

$$\|e^{-\alpha u}\|_{L^1(K)} \leq A$$

for every function u in $\mathscr{F}$.

P.s.h. functions are in general unbounded. However, the last result shows that such functions are nearly bounded. The above family $\mathscr{F}$ is uniformly bounded from above on K. So, we also have the estimate

$$\|e^{\alpha |u|}\|_{L^1(K)} \leq A$$

for u in $\mathscr{F}$ and for some (other) constants α, A. More precise estimates can be obtained in terms of the maximal Lelong number of $dd^c u$ in a neighbourhood of K.

Define the *Lelong number* $\nu(u, a)$ of u at a as the Lelong number of $dd^c u$ at a. The following result describes the relation with the singularity of p.s.h. functions near a pole. We fix here a local coordinate system for X.

Proposition A.23. *The Lelong number $\nu(u, a)$ is the supremum of the number ν such that the inequality $u(z) \leq \nu \log \|z - a\|$ holds in a neighbourhood of a.*

If S is a positive closed (p, p)-current, the Lelong number $\nu(S, a)$ can be computed as the mass at a of the measure $S \wedge (dd^c \log \|z - a\|)^{k-p}$. This property allows to prove the following result, due to Demailly [DEM], which is useful in dynamics.

Proposition A.24. *Let $\tau : (\mathbb{C}^k, 0) \to (\mathbb{C}^k, 0)$ be a germ of an open holomorphic map with $\tau(0) = 0$. Let d denote the multiplicity of τ at 0. Let S be a positive closed (p, p)-current on a neighbourhood of 0. Then, the Lelong number of $\tau_*(S)$ at 0 satisfies the inequalities*

$$\nu(S, 0) \leq \nu(\tau_*(S), 0) \leq d^{k-p} \nu(S, 0).$$

In particular, we have $\nu(\tau_*(S), 0) = 0$ if and only if $\nu(S, 0) = 0$.

Assume now that X is a compact Kähler manifold and ω is a Kähler form on X. If S is a dd^c-closed (p, p)-current, we can, using the dd^c-lemma, define a linear form on $H^{k-p,k-p}(X, \mathbb{C})$ by $[\alpha] \mapsto \langle S, \alpha \rangle$. Therefore, the Poincaré duality implies that S is canonically associated to a class $[S]$ in $H^{p,p}(X, \mathbb{C})$. If S is real then $[S]$ is in $H^{p,p}(X, \mathbb{R})$. If S is positive, its mass $\langle S, \omega^{k-p} \rangle$ depends only on the class $[S]$. So, the mass of positive dd^c-closed currents can be computed cohomologically. In $\mathbb{P}^k$, the mass of ω^k_{FS} is 1 since ω^k_{FS} is a probability measure. If H is a subspace of codimension p of $\mathbb{P}^k$, then the current associated to H is of mass 1 and it belongs to the class $[\omega^p_{FS}]$. If Y is an analytic set of pure codimension p of $\mathbb{P}^k$, *the degree* $\deg(Y)$ of Y is by definition the number of points in its intersection with a generic projective space of dimension p. One can check that the cohomology class of Y is $\deg(Y)[\omega^p_{FS}]$. The volume of Y, obtained using Wirtinger's theorem A.2, is equal to $\frac{1}{p!} \deg(Y)$.

Exercise A.25. With the notation of Exercise A.11, show that $\tau_*(\varphi)$ is p.s.h. if φ is p.s.h.

Exercise A.26. Using that $\nu(S,a,r)$ is decreasing, show that if (S_n) is a sequence of positive closed (p,p)-currents on X converging to a current S and (a_n) is a sequence in X converging to a, then $\limsup \nu(S_n,a_n) \leq \nu(S,a)$.

Exercise A.27. Let S and S' be positive closed $(1,1)$-currents such that $S' \leq S$. Assume that the local potentials of S are bounded or continuous. Show that the local potentials of S' are also bounded or continuous.

Exercise A.28. Let $\mathscr{F}$ be an L^1_{loc} bounded family of p.s.h. functions on X. Let K be a compact subset of X. Show that $\mathscr{F}$ is locally bounded from above and that there is $c > 0$ such that $\|dd^c u\|_K \leq c$ for every $u \in \mathscr{F}$. Prove that there is a constant $\nu > 0$ such that $\nu(u,a) \leq \nu$ for $u \in \mathscr{F}$ and $a \in K$.

Exercise A.29. Let Y_i, $1 \leq i \leq m$, be analytic sets of pure codimension p_i in $\mathbb{P}^k$. Assume $p_1 + \cdots + p_m \leq k$. Show that the intersection of the Y_i's is a non-empty analytic set of dimension $\geq k - p_1 - \cdots - p_m$.

A.3 Intersection, Pull-back and Slicing

We have seen that positive closed currents generalize differential forms and analytic sets. However, it is not always possible to extend the calculus on forms or on analytic sets to currents. We will give here some results which show how positive closed currents are flexible and how they are rigid.

The theory of intersection is much more developed in bidegree $(1,1)$ thanks to the use of their potentials which are p.s.h. functions. The case of continuous potentials was considered by Chern-Levine-Nirenberg [CLN]. Bedford-Taylor [BD] developed a nice theory when the potentials are locally bounded. The case of unbounded potentials was considered by Demailly [DE] and Fornæss-Sibony [FS2, S2]. We have the following general definition.

Let S be a positive closed (p,p)-current on X with $p \leq k-1$. If ω is a fixed Hermitian form on X as above, then $S \wedge \omega^{k-p}$ is a positive measure which is called *the trace measure* of S. In local coordinates, the coefficients of S are measures, bounded by a constant times the trace measure. Now, if u is a p.s.h function on X, locally integrable with respect to the trace measure of S, then uS is a current on X and we can define

$$dd^c u \wedge S := dd^c(uS).$$

Since u can be locally approximated by decreasing sequences of smooth p.s.h. functions, it is easy to check that the previous intersection is a positive closed $(p+1,p+1)$-current with support contained in supp(S). When u is pluriharmonic, $dd^c u \wedge S$ vanishes identically. So, the intersection depends only on $dd^c u$ and on S. If R is a positive closed $(1,1)$-current on X, one defines $R \wedge S$ as above using local

potentials of R. In general, $dd^c u \wedge S$ does not depend continuously on u and S. The following proposition is a consequence of Hartogs' lemma.

Proposition A.30. *Let $u^{(n)}$ be p.s.h. functions on X which converge in the Hartogs' sense to a p.s.h. function u. If u is locally integrable with respect to the trace measure of S, then $dd^c u^{(n)} \wedge S$ are well-defined and converge to $dd^c u \wedge S$. If u is continuous and S_n are positive closed $(1,1)$-currents converging to S, then $dd^c u^{(n)} \wedge S_n$ converge to $dd^c u \wedge S$.*

If $u_1, \ldots, u_q$, with $q \le k - p$, are p.s.h. functions, we can define by induction the wedge-product

$$dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge S$$

when some integrability conditions are satisfied, for example when the u_i are locally bounded. In particular, if $u_j^{(n)}$, $1 \le j \le q$, are continuous p.s.h. functions converging locally uniformly to continuous p.s.h. functions u_j and if S_n are positive closed converging to S, then

$$dd^c u_1^{(n)} \wedge \ldots \wedge dd^c u_q^{(n)} \wedge S_n \to dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge S$$

The following version of the Chern-Levine-Nirenberg inequality is a very useful result [CLN, DEM].

Theorem A.31. *Let S be a positive closed (p,p)-current on X. Let $u_1, \ldots, u_q$, $q \le k - p$, be locally bounded p.s.h. functions on X and K a compact subset of X. Then there is a constant $c > 0$ depending only on K and X such that if v is p.s.h. on X then*

$$\|v dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge S\|_K \le c \|v\|_{L^1(\sigma_S)} \|u_1\|_{L^\infty(X)} \ldots \|u_q\|_{L^\infty(X)},$$

where σ_S denotes the trace measure of S.

This inequality implies that p.s.h. functions are locally integrable with respect to the current $dd^c u_1 \wedge \ldots \wedge dd^c u_q$. We deduce the following corollary.

Corollary A.32. *Let $u_1, \ldots, u_p$, $p \le k$, be locally bounded p.s.h. functions on X. Then, the current $dd^c u_1 \wedge \ldots \wedge dd^c u_p$ has no mass on locally pluripolar sets, in particular on proper analytic sets of X.*

We give now two other regularity properties of the wedge-product of currents with Hölder continuous local potentials.

Proposition A.33. *Let S be a positive closed (p,p)-current on X and q a positive integer such that $q \le k - p$. Let u_i be Hölder continuous p.s.h. functions of Hölder exponents α_i with $0 < \alpha_i \le 1$ and $1 \le i \le q$. Then, the current $dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge S$ has no mass on Borel sets with Hausdorff dimension less than or equal to $2(k - p - q) + \alpha_1 + \cdots + \alpha_q$.*

The proof of this result is given in [S3]. It is based on a mass estimate on a ball in term of the radius which is a consequence of the Chern-Levine-Nirenberg inequality.

We say that a positive measure ν in X is *locally moderate* if for any compact subset K of X and any compact family $\mathscr{F}$ of p.s.h. functions in a neighbourhood of K, there are positive constants α and c such that

$$\int_K e^{-\alpha u} d\nu \le c$$

for u in $\mathscr{F}$. This notion was introduced in [DS1]. We say that a positive current is *locally moderate* if its trace measure is locally moderate. The following result was obtained in [DNS].

Theorem A.34. *Let S be a positive closed (p, p)-current on X and u a p.s.h. function on X. Assume that S is locally moderate and u is Hölder continuous. Then the current $dd^c u \wedge S$ is locally moderate. In particular, wedge-products of positive closed $(1, 1)$-currents with Hölder continuous local potentials are locally moderate.*

Theorem A.22 implies that a measure defined by a smooth form is locally moderate. Theorem A.34 implies, by induction, that $dd^c u_1 \wedge \ldots \wedge dd^c u_p$ is locally moderate when the p.s.h. functions u_j are Hölder continuous. So, using p.s.h. functions as test functions, the previous currents satisfy similar estimates as smooth forms do. One may also consider that Theorem A.34 strengthens A.22 and gives a strong compactness property for p.s.h. functions. The estimate has many consequences in complex dynamics.

The proof of Theorem A.34 is based on a mass estimate of $dd^c u \wedge S$ on the sub-level set $\{v < -M\}$ of a p.s.h function v. Some estimates are easily obtained for u continuous using the Chern-Levine-Nirenberg inequality or for u of class $\mathscr{C}^2$. The case of Hölder continuous function uses arguments close to the interpolation between the Banach spaces $\mathscr{C}^0$ and $\mathscr{C}^2$. However, the non-linearity of the estimate and the positivity of currents make the problem more subtle.

We discuss now the pull-back of currents by holomorphic maps which are not submersions. The problem can be considered as a particular case of the general intersection theory, but we will not discuss this point here. The following result was obtained in [DS8].

Theorem A.35. *Let $\tau : X' \to X$ be an open holomorphic map between complex manifolds of the same dimension. Then the pull-back operator τ^* on smooth positive closed (p, p)-forms can be extended in a canonical way to a continuous operator on positive closed (p, p)-currents S on X. If S has no mass on a Borel set $K \subset X$, then $\tau^*(S)$ has no mass on $\tau^{-1}(K)$. The result also holds for negative currents S such that $dd^c S$ is positive.*

By canonical way, we mean that the extension is functorial. More precisely, one can locally approximate S by a sequence of smooth positive closed forms. The pull-back of these forms converge to some positive closed (p, p)-current which does not depend on the chosen sequence of forms. This limit defines the pull-back

current $\tau^*(S)$. The result still holds when X' is singular. In the case of bidegree $(1,1)$, we have the following result due to Méo [ME].

Proposition A.36. *Let* $\tau : X' \to X$ *be a holomorphic map between complex manifolds. Assume that* τ *is dominant, that is, the image of* τ *contains an open subset of* X. *Then the pull-back operator* τ^* *on smooth positive closed* $(1,1)$-*forms can be extended in a canonical way to a continuous operator on positive closed* $(1,1)$-*currents* S *on* X.

Indeed, locally we can write $S = dd^c u$ with u p.s.h. The current $\tau^*(S)$ is then defined by $\tau^*(S) := dd^c(u \circ \tau)$. One can check that the definition does not depend on the choice of u.

The remaining part of this paragraph deals with the slicing of currents. We only consider a situation used in this course. Let $\pi : X \to V$ be a dominant holomorphic map from X to a manifold V of dimension l and S a current on X. Slicing theory allows to define the slice $\langle S, \pi, \theta \rangle$ of some currents S on X by the fiber $\pi^{-1}(\theta)$. Slicing theory generalizes the restriction of forms to fibers. One can also consider it as a generalization of Sard's and Fubini's theorems for currents or as a special case of intersection theory: the slice $\langle S, \pi, \theta \rangle$ can be seen as the wedge-product of S with the current of integration on $\pi^{-1}(\theta)$. We can consider the slicing of $\mathbb{C}$-flat currents, in particular, of (p,p)-currents such that S and $dd^c S$ are of order 0. The operation preserves positivity and commutes with ∂, $\overline{\partial}$. If φ is a smooth form on X then $\langle S \wedge \varphi, \pi, \theta \rangle = \langle S, \pi, \theta \rangle \wedge \varphi$. Here, we only consider positive closed $(k-l,k-l)$-currents S. In this case, the slices $\langle S, \pi, \theta \rangle$ are positive measures on X with support in $\pi^{-1}(\theta)$.

Let y denote the coordinates in a chart of V and $\lambda_V := (dd^c \|y\|^2)^l$ the Euclidean volume form associated to y. Let $\psi(y)$ be a positive smooth function with compact support such that $\int \psi \lambda_V = 1$. Define $\psi_\varepsilon(y) := \varepsilon^{-2l} \psi(\varepsilon^{-1} y)$ and $\psi_{\theta,\varepsilon}(y) := \psi_\varepsilon(y - \theta)$. The measures $\psi_{\theta,\varepsilon} \lambda_V$ approximate the Dirac mass at θ. For every smooth test function Φ on X, we have

$$\langle S, \pi, \theta \rangle(\Phi) = \lim_{\varepsilon \to 0} \langle S \wedge \pi^*(\psi_{\theta,\varepsilon} \lambda_V), \Phi \rangle$$

when $\langle S, \pi, \theta \rangle$ exists. This property holds for all choice of ψ. Conversely, when the previous limit exists and is independent of ψ, it defines the measure $\langle S, \pi, \theta \rangle$ and we say that $\langle S, \pi, \theta \rangle$ *is well-defined*. The slice $\langle S, \pi, \theta \rangle$ is well-defined for θ out of a set of Lebesgue measure zero in V and the following formula holds for smooth forms Ω of maximal degree with compact support in V:

$$\int_{\theta \in V} \langle S, \pi, \theta \rangle(\Phi) \Omega(\theta) = \langle S \wedge \pi^*(\Omega), \Phi \rangle.$$

We recall the following result which was obtained in [DS7].

Theorem A.37. *Let* V *be a complex manifold of dimension* l *and let* π *denote the canonical projection from* $\mathbb{C}^k \times V$ *onto* V. *Let* S *be a positive closed current of bidimension* (l,l) *on* $\mathbb{C}^k \times V$, *supported on* $K \times V$ *for a given compact subset* K

of $\mathbb{C}^k$. Then the slice $\langle S, \pi, \theta \rangle$ is well-defined for every θ in V and is a positive measure whose mass is independent of θ. Moreover, if Φ is a p.s.h. function in a neighbourhood of supp(S)*, then the function $\theta \mapsto \langle S, \pi, \theta \rangle (\Phi)$ is p.s.h.*

The mass of $\langle S, \pi, \theta \rangle$ is called the *slice mass* of S. The set of currents S as above with bounded slice mass is compact for the weak topology on currents. In particular, their masses are locally uniformly bounded on $\mathbb{C}^k \times V$. In general, the slice $\langle S, \pi, \theta \rangle$ does not depend continuously on S nor on θ. The last property in Theorem A.37 shows that the dependence on θ satisfies a semi-continuity property. More generally, we have that $(\theta, S) \mapsto \langle S, \pi, \theta \rangle (\Phi)$ is upper semi-continuous for Φ p.s.h. We deduce easily from the above definition that the slice mass of S depends continuously on S.

Exercise A.38. Let X, X' be complex manifolds. Let ν be a positive measure with compact support on X such that p.s.h. functions on X are ν-integrable. If u is a p.s.h. function on $X \times X'$, show that $x' \mapsto \int u(x, x') d\nu(x)$ is a p.s.h function on X'. Show that if ν, ν' are positive measures on X, X' which are locally moderate, then $\nu \otimes \nu'$ is a locally moderate measure on $X \times X'$.

Exercise A.39. Let S be a positive closed $(1,1)$-current on the unit ball of $\mathbb{C}^k$. Let $\pi : \widehat{\mathbb{C}^k} \to \mathbb{C}^k$ be the blow-up of $\mathbb{C}^k$ at 0 and E the exceptional set. Show $\pi^*(S)$ is equal to $\nu[E] + S'$, where ν is the Lelong number of S at 0 and S' is a current without mass on E.

A.4 Currents on Projective Spaces

In this paragraph, we will introduce quasi-potentials of currents, the spaces of d.s.h. functions, DSH currents and the complex Sobolev space which are used as observables in complex dynamics. We also introduce PB, PC currents and the notion of super-potentials which are crucial in the calculus with currents in higher bidegree.

Recall that the Fubini-Study form ω_{FS} on $\mathbb{P}^k$ satisfies $\int_{\mathbb{P}^k} \omega_{FS}^k = 1$. If S is a positive closed (p,p)-current, the mass of S is given by by $\|S\| := \langle S, \omega_{FS}^{k-p} \rangle$. Since $H^{p,p}(\mathbb{P}^k, \mathbb{R})$ is generated by ω_{FS}^p, such a current S is cohomologous to $c\omega_{FS}^p$ where c is the mass of S. So, $S - c\omega_{FS}^p$ is exact and the dd^c-lemma, which also holds for currents, implies that there exists a $(p-1, p-1)$-current U, such that $S = c\omega_{FS}^p + dd^cU$. We call U *a quasi-potential* of S. We have in fact the following more precise result [DS10].

Theorem A.40. *Let S be a positive closed (p,p)-current of mass 1 in $\mathbb{P}^k$. Then, there is a negative form U such that $dd^cU = S - \omega_{FS}^p$. For r, s with $1 \le r < k/(k-1)$ and $1 \le s < 2k/(2k-1)$, we have*

$$\|U\|_{L^r} \le c_r \quad and \quad \|\nabla U\|_{L^s} \le c_s,$$

where c_r, c_s are constants independent of S. Moreover, U depends linearly and continuously on S with respect to the weak topology on the currents S and the L^r topology on U.

The construction of U uses a kernel constructed in Bost-Gillet-Soulé [BGS]. We call U *the Green quasi-potential* of S. When $p = 1$, two quasi-potentials of S differ by a constant. So, the solution is unique if we require that $\langle \omega_{FS}^k, U \rangle = 0$. In this case, we have a bijective and bi-continuous correspondence $S \leftrightarrow u$ between positive closed $(1,1)$-currents S and their normalized quasi-potentials u.

By maximum principle, p.s.h. functions on a compact manifold are constant. However, the interest of p.s.h. functions is their type of local singularities. S.T. Yau introduced in [YA] the useful notion of quasi-p.s.h. functions. A *quasi-p.s.h. function* is locally the difference of a p.s.h. function and a smooth one. Several properties of quasi-p.s.h. functions can be deduced from properties of p.s.h. functions. If u is a quasi-p.s.h. function on $\mathbb{P}^k$ there is a constant $c > 0$ such that $dd^c u \geq -c\omega_{FS}$. So, $dd^c u$ is the difference of a positive closed $(1,1)$-current and a smooth positive closed $(1,1)$-form: $dd^c u = (dd^c u + c\omega_{FS}) - c\omega_{FS}$. Conversely, if S is a positive closed $(1,1)$-current cohomologous to a real $(1,1)$-form α, there is a quasi-p.s.h. function u, unique up to a constant, such that $dd^c u = S - \alpha$. The following proposition is easily obtained using a convolution on the group of automorphisms of $\mathbb{P}^k$, see Demailly [DEM] for analogous results on compact Kähler manifolds.

Proposition A.41. *Let u be a quasi-p.s.h. function on $\mathbb{P}^k$ such that $dd^c u \geq -\omega_{FS}$. Then, there is a sequence (u_n) of smooth quasi-p.s.h. functions decreasing to u such that $dd^c u_n \geq -\omega_{FS}$. In particular, if S is a positive closed $(1,1)$-current on $\mathbb{P}^k$, then there are smooth positive closed $(1,1)$-forms S_n converging to S.*

A subset E of $\mathbb{P}^k$ is *pluripolar* if it is contained in $\{u = -\infty\}$ where u is a quasi-p.s.h. function. It is *complete pluripolar* if there is a quasi-p.s.h. function u such that $E = \{u = -\infty\}$. It is easy to check that analytic sets are complete pluripolar and that a countable union of pluripolar sets is pluripolar. The following capacity is close to a notion of capacity introduced by H. Alexander in [AL]. The interesting point here is that our definition extends to general compact Kähler manifold [DS6]. We will see that the same idea allows to define the capacity of a current. Let $\mathscr{P}_1$ denote the set of quasi-p.s.h. functions u on $\mathbb{P}^k$ such that $\max_{\mathbb{P}^k} u = 0$. *The capacity* of a Borel set E in $\mathbb{P}^k$ is

$$\operatorname{cap}(E) := \inf_{u \in \mathscr{P}_1} \exp\left(\sup_E u\right).$$

The Borel set E is pluripolar if and only if $\operatorname{cap}(E) = 0$. It is not difficult to show that when the volume of E tends to the volume of $\mathbb{P}^k$ then $\operatorname{cap}(E)$ tends to 1.

The space of d.s.h. functions (differences of quasi-p.s.h. functions) and the complex Sobolev space of functions on compact Kähler manifolds were introduced by the authors in [DS6, DS11]. They satisfy strong compactness properties and are invariant under the action of holomorphic maps. Using them as test functions, permits to obtain several results in complex dynamics.

A function on $\mathbb{P}^k$ is called *d.s.h.* if it is equal outside a pluripolar set to the difference of two quasi-p.s.h. functions. We identify two d.s.h. functions if they are equal outside a pluripolar set. Let $\mathrm{DSH}(\mathbb{P}^k)$ denote the space of d.s.h. functions on $\mathbb{P}^k$. We deduce easily from properties of p.s.h. functions that $\mathrm{DSH}(\mathbb{P}^k)$ is contained

in $L^p(\mathbb{P}^k)$ for $1 \leq p < \infty$. If u is d.s.h. then $dd^c u$ can be written as the difference of two positive closed $(1,1)$-currents which are cohomologous. Conversely, if $S^\pm$ are positive closed $(1,1)$-currents of the same mass, then there is a d.s.h. function u, unique up to a constant, such that $dd^c u = S^+ - S^-$.

We introduce several equivalent norms on $\mathrm{DSH}(\mathbb{P}^k)$. Define

$$\|u\|_{\mathrm{DSH}} := |\langle \omega_{\mathrm{FS}}^k, u \rangle| + \min \|S^\pm\|,$$

where the minimum is taken over positive closed $(1,1)$-currents $S^\pm$ such that $dd^c u = S^+ - S^-$. The term $|\langle \omega_{\mathrm{FS}}^k, u \rangle|$ may be replaced with $\|u\|_{L^p}$, $1 \leq p < \infty$; we then obtain equivalent norms. The space of d.s.h. functions endowed with the above norm is a Banach space. However, we will use on this space *a weaker topology*: we say that a sequence (u_n) converges to u in $\mathrm{DSH}(\mathbb{P}^k)$ if u_n converge to u in the sense of currents and if (u_n) is bounded with respect to $\|\cdot\|_{\mathrm{DSH}}$. Under the last condition on the DSH-norm, the convergence in the sense of currents of u_n is equivalent to the convergence in L^p for $1 \leq p < \infty$. We have the following proposition [DS6].

Proposition A.42. *Let u be a d.s.h. function on $\mathbb{P}^k$ such that $\|u\|_{\mathrm{DSH}} \leq 1$. Then there are negative quasi-p.s.h. function $u^\pm$ such that $u = u^+ - u^-$, $\|u^\pm\|_{\mathrm{DSH}} \leq c$ and $dd^c u^\pm \geq -c\omega_{\mathrm{FS}}$, where $c > 0$ is a constant independent of u.*

A positive measure on $\mathbb{P}^k$ is said to be *PC* [10] if it can be extended to a continuous linear form on $\mathrm{DSH}(\mathbb{P}^k)$. Here, the continuity is with respect to the weak topology on d.s.h. functions. A positive measure is *PB* [11] if quasi-p.s.h. functions are integrable with respect to this measure. PB measures have no mass on pluripolar sets and d.s.h. functions are integrable with respect to such measures. PC measures are always PB. Let   be a non-zero PB positive measure on X. Define

$$\|u\| := |\langle   , u \rangle| + \min \|S^\pm\|,$$

with $S^\pm$ as above. We have the following useful property [DS6].

Proposition A.43. *The semi-norm $\|\cdot\|$ is in fact a norm on $\mathrm{DSH}(\mathbb{P}^k)$ which is equivalent to $\|\cdot\|_{\mathrm{DSH}}$.*

One can extend the above notions to currents but the definitions are slightly different. Let $\mathrm{DSH}^p(\mathbb{P}^k)$ denote the space generated by negative (p,p)-currents Φ such that $dd^c \Phi$ is the difference of two positive closed $(p+1, p+1)$-currents. A DSH (p,p)-current, i.e. a current in $\mathrm{DSH}^p(\mathbb{P}^k)$, is not an L^1 form in general. Define the $\|\Phi\|_{\mathrm{DSH}}$-norm of a negative current Φ in $\mathrm{DSH}^p(\mathbb{P}^k)$ by

$$\|\Phi\|_{\mathrm{DSH}} := \|\Phi\| + \min \|\Omega^\pm\|,$$

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[10] In dimension 1, the measure is PC if and only if its local Potentials are Continuous.

[11] In dimension 1, the measure is PB if and only if its local Potentials are Bounded.

where $\Omega^{\pm}$ are positive closed such that $dd^c\Phi = \Omega^+ - \Omega^-$. For a general Φ in $\mathrm{DSH}^p(\mathbb{P}^k)$ define

$$\|\Phi\|_{\mathrm{DSH}} := \min(\|\Phi^+\|_{\mathrm{DSH}} + \|\Phi^-\|_{\mathrm{DSH}}),$$

where $\Phi^{\pm}$ are negative currents in $\mathrm{DSH}^p(X)$ such that $\Phi = \Phi^+ - \Phi^-$. We also consider on this space *the weak topology*: a sequence (Φ_n) converges to Φ in $\mathrm{DSH}^p(\mathbb{P}^k)$ if it converges to Φ in the sense of currents and if $(\|\Phi_n\|_{\mathrm{DSH}})$ is bounded. Using a convolution on the group of automorphisms of $\mathbb{P}^k$, we can show that smooth forms are dense in $\mathrm{DSH}^p(\mathbb{P}^k)$.

A positive closed (p,p)-current S is called *PB* if there is a constant $c > 0$ such that $|\langle S, \Phi \rangle| \leq c\|\Phi\|_{\mathrm{DSH}}$ for any real smooth $(k-p, k-p)$-form Φ. The current S is *PC* if it can be extended to a linear continuous form on $\mathrm{DSH}^{k-p}(\mathbb{P}^k)$. The continuity is with respect to the weak topology we consider on $\mathrm{DSH}^{k-p}(\mathbb{P}^k)$. PC currents are PB. We will see that these notions correspond to currents with bounded or continuous super-potentials. As a consequence of Theorem A.35, we have the following useful result.

Proposition A.44. *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic surjective map. Then, the operator f^* on smooth forms has a continuous extension $f^* : \mathrm{DSH}^p(\mathbb{P}^k) \to \mathrm{DSH}^p(\mathbb{P}^k)$. If S is a current on $\mathrm{DSH}^p(\mathbb{P}^k)$ with no mass on a Borel set A, then $f^*(S)$ has no mass on $f^{-1}(A)$.*

Another useful functional space is *the complex Sobolev space $W^*(\mathbb{P}^k)$*. Its definition uses the complex structure of $\mathbb{P}^k$. In dimension one, $W^*(\mathbb{P}^1)$ coincides with the Sobolev space $W^{1,2}(\mathbb{P}^1)$ of real-valued functions in L^2 with gradient in L^2. In higher dimension, $W^*(\mathbb{P}^k)$ is the space of functions u in $W^{1,2}(\mathbb{P}^k)$ such that $i\partial u \wedge \overline{\partial} u$ is bounded by a positive closed $(1,1)$-current Θ. We define

$$\|u\|_{W^*} := |\langle \omega_{\mathrm{FS}}^k, u \rangle| + \min \|\Theta\|^{1/2}$$

with Θ as above, see [DS11, V3]. By Sobolev-Poincaré inequality, the term $|\langle \omega_{\mathrm{FS}}^k, u \rangle|$ may be replaced with $\|u\|_{L^1}$ or $\|u\|_{L^2}$; we then obtain equivalent norms. *The weak topology* on $W^*(\mathbb{P}^k)$ is defined as in the case of d.s.h. functions: a sequence (u_n) converges in $W^*(\mathbb{P}^k)$ to a function u if it converges to u in the sense of currents and if $(\|u_n\|_{W^*})$ is bounded. A positive measure   is *WPC* if it can be extended to a linear continuous form on $W^*(\mathbb{P}^k)$. If u is a strictly negative quasi-p.s.h. function, one can prove that $\log(-u)$ is in $W^*(\mathbb{P}^k)$. This allows to show that WPC measures have no mass on pluripolar sets.

In the rest of the paragraph, we will introduce the notion of super-potentials associated to positive closed (p,p)-currents. They are canonical functions defined on infinite dimensional spaces and are, in some sense, quasi-p.s.h. functions there. Super-potentials were introduced by the authors in order to replace ordinary quasi-p.s.h. functions which are used as quasi-potentials for currents of bidegree $(1,1)$. The theory is satisfactory in the case of projective spaces [DS10] and can be easily extended to homogeneous manifolds.

Let $\mathscr{C}_{k-p+1}(\mathbb{P}^k)$ denote the convex set of positive closed currents of bidegree $(k-p+1,k-p+1)$ and of mass 1, i.e. currents cohomologous to $\omega_{\mathrm{FS}}^{k-p+1}$. Let S be a positive closed (p,p)-current on $\mathbb{P}^k$. We assume for simplicity that S is of mass 1; the general case can be deduced by linearity. The super-potential[12] $\mathscr{U}_S$ of S is a function on $\mathscr{C}_{k-p+1}(\mathbb{P}^k)$ with values in $\mathbb{R} \cup \{-\infty\}$. Let R be a current in $\mathscr{C}_{k-p+1}(\mathbb{P}^k)$ and U_R a potential of $R - \omega_{\mathrm{FS}}^{k-p+1}$. Subtracting from U_R a constant times $\omega_{\mathrm{FS}}^{k-p}$ allows to have $\langle U_R, \omega_{\mathrm{FS}}^p \rangle = 0$. We say that U_R is a quasi-potential of mean 0 of R. Formally, i.e. in the case where R and U_R are smooth, the value of $\mathscr{U}_S$ at R is defined by

$$\mathscr{U}_S(R) := \langle S, U_R \rangle.$$

One easily check using Stokes' formula that formally if U_S is a quasi-potential of mean 0 of S, then $\mathscr{U}_S(R) = \langle U_S, R \rangle$. Therefore, the previous definition does not depend on the choice of U_R or U_S. By definition, we have $\mathscr{U}_S(\omega_{\mathrm{FS}}^{k-p+1}) = 0$. Observe also that when S is smooth, the above definition makes sense for every R and $\mathscr{U}_S$ is a continuous affine function on $\mathscr{C}_{k-p+1}(\mathbb{P}^k)$. It is also clear that if $\mathscr{U}_S = \mathscr{U}_{S'}$, then $S = S'$. The following theorem allows to define $\mathscr{U}_S$ in the general case.

Theorem A.45. *The above function $\mathscr{U}_S$, which is defined on smooth forms R in $\mathscr{C}_{k-p+1}(\mathbb{P}^k)$, can be extended to an affine function on $\mathscr{C}_{k-p+1}(\mathbb{P}^k)$ with values in $\mathbb{R} \cup \{-\infty\}$ by*

$$\mathscr{U}_S(R) := \limsup \mathscr{U}_S(R'),$$

where R' is smooth in $\mathscr{C}_{k-p+1}(\mathbb{P}^k)$ and converges to R. We have $\mathscr{U}_S(R) = \mathscr{U}_R(S)$. Moreover, there are smooth positive closed (p,p)-forms S_n of mass 1 and constants c_n converging to 0 such that $\mathscr{U}_{S_n} + c_n$ decrease to $\mathscr{U}_S$. In particular, $\mathscr{U}_{S_n}$ converge pointwise to $\mathscr{U}_S$.

For bidegree $(1,1)$, there is a unique quasi-p.s.h. function u_S such that $dd^c u_S = S - \omega_{\mathrm{FS}}$ and $\langle \omega_{\mathrm{FS}}^k, u_S \rangle = 0$. If δ_a denotes the Dirac mass at a, we have $\mathscr{U}_S(\delta_a) = u_S(a)$. Dirac masses are extremal elements in $\mathscr{C}_k(\mathbb{P}^k)$. The super-potential $\mathscr{U}_S$ in this case is just the affine extension of u_S, that is, we have for any probability measure v:

$$\mathscr{U}_S(v) = \int \mathscr{U}_S(\delta_a) dv(a) = \int u_S(a) dv(a).$$

The function $\mathscr{U}_S$ extends the action $\langle S, \Phi \rangle$ on smooth forms Φ to $\langle S, U \rangle$ where U is a quasi-potential of a positive closed current. Super-potentials satisfy analogous properties as quasi-p.s.h. functions do. They are upper semi-continuous and bounded from above by a universal constant. Note that we consider here the weak topology on $\mathscr{C}_{k-p+1}(\mathbb{P}^k)$. We have the following version of the Hartogs' lemma.

Proposition A.46. *Let S_n be positive closed (p,p)-currents of mass 1 on $\mathbb{P}^k$ converging to S. Then for every continuous function $\mathscr{U}$ on $\mathscr{C}_{k-p+1}$ with $\mathscr{U}_S < \mathscr{U}$, we have $\mathscr{U}_{S_n} < \mathscr{U}$ for n large enough. In particular, $\limsup \mathscr{U}_{S_n} \le \mathscr{U}_S$.*

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[12] The super-potential we consider here corresponds to the super-potential of mean 0 in [DS10]. The other super-potentials differ from $\mathscr{U}_S$ by constants.

We say that S_n converge to S *in the Hartogs' sense* if S_n converge to S and if there are constants c_n converging to 0 such that $\mathcal{U}_{S_n} + c_n \geq \mathcal{U}_S$. If $\mathcal{U}_{S_n}$ converge uniformly to $\mathcal{U}_S$, we say that S_n *converge SP-uniformly* to S.

One can check that PB and PC currents correspond to currents of bounded or continuous super-potential. In the case of bidegree $(1,1)$, they correspond to currents with bounded or continuous quasi-potential. We say that S' is *more diffuse than* S if $\mathcal{U}_{S'} - \mathcal{U}_S$ is bounded from below. So, PB currents are more diffuse than any other currents.

In order to prove the above results and to work with super-potentials, we have to consider a geometric structure on $\mathscr{C}_{k-p+1}(\mathbb{P}^k)$. In a weak sense, $\mathscr{C}_{k-p+1}(\mathbb{P}^k)$ can be seen as a space of infinite dimension which contains many "analytic" sets of finite dimension that we call *structural varieties*. Let V be a complex manifold and $\mathscr{R}$ a positive closed current of bidegree $(k-p+1, k-p+1)$ on $V \times \mathbb{P}^k$. Let π_V denote the canonical projection map from $V \times \mathbb{P}^k$ onto V. One can prove that the slice $\langle \mathscr{R}, \pi_V, \theta \rangle$ is defined for θ outside a locally pluripolar set of V. Each slice can be identified with a positive closed (p,p)-current R_θ on $\mathbb{P}^k$. Its mass does not depend on θ. So, multiplying $\mathscr{R}$ with a constant, we can assume that all the R_θ are in $\mathscr{C}_{k-p+1}(\mathbb{P}^k)$. The map $\tau(\theta) := R_\theta$ or the family (R_θ) is called *a structural variety* of $\mathscr{C}_{k-p+1}(\mathbb{P}^k)$. The restriction of $\mathcal{U}_S$ to this structural variety, i.e. $\mathcal{U}_S \circ \tau$, is locally a d.s.h. function or identically $-\infty$. When the structural variety is nice enough, this restriction is quasi-p.s.h. or identically $-\infty$. In practice, we often use some special structural discs parametrized by θ in the unit disc of $\mathbb{C}$. They are obtained by convolution of a given current R with a smooth probability measure on the group $\mathrm{PGL}(\mathbb{C}, k+1)$ of automorphisms of $\mathbb{P}^k$.

Observe that since the correspondence $S \leftrightarrow \mathcal{U}_S$ is $1:1$, the compactness on positive closed currents should induce some compactness on super-potentials. We have the following result.

Theorem A.47. *Let $W \subset \mathbb{P}^k$ be an open set and $K \subset W$ a compact set. Let S be a current in $\mathscr{C}_p(\mathbb{P}^k)$ with support in K and R a current in $\mathscr{C}_{k-p+1}(\mathbb{P}^k)$. Assume that the restriction of R to W is a bounded form. Then, the super-potential $\mathcal{U}_S$ of S satisfies*

$$|\mathcal{U}_S(R)| \leq A\big(1 + \log^+ \|R\|_{\infty,W}\big)$$

where $A > 0$ is a constant independent of S, R and $\log^+ := \max(0, \log)$.

This result can be applied to $K = W = \mathbb{P}^k$ and can be considered as a version of the exponential estimate in Theorem A.22. Indeed, the weaker estimate $|\mathcal{U}_S(R)| \lesssim 1 + \|R\|_\infty$ is easy to obtain. It corresponds to the L^1 estimate on the quasi-p.s.h. function u_S in the case of bidegree $(1,1)$.

Using the analogy with the bidegree $(1,1)$ case, we define *the capacity* of a current R as

$$\mathrm{cap}(R) := \inf_S \exp\big(\mathcal{U}_S(R) - \max \mathcal{U}_S\big).$$

This capacity describes rather the regularity of R: an R with big capacity is somehow more regular. Theorem A.47 implies that $\mathrm{cap}(R) \gtrsim \|R\|_\infty^{-\lambda}$ for some universal constant $\lambda > 0$. This property is close to the capacity estimate for Borel sets in term of volume.

Super-potentials allow to develop a theory of intersection of currents in higher bidegree. Here, the fact that $\mathcal{U}_S$ has a value at every point (i.e. at every current $R \in \mathcal{C}_{k-p+1}(\mathbb{P}^k)$) is crucial. Let S, S' be positive closed currents of bidegree (p,p) and (p',p') with $p+p' \leq k$. We assume for simplicity that their masses are equal to 1. We say that S and S' are *wedgeable* if $\mathcal{U}_S$ is finite at $S' \wedge \omega_{FS}^{k-p-p'+1}$. This property is symmetric on S and S'. If $\widetilde{S}, \widetilde{S}'$ are more diffuse than S, S' and if S, S' are wedgeable, then $\widetilde{S}, \widetilde{S}'$ are wedgeable.

Let Φ be a real smooth form of bidegree $(k-p-p', k-p-p')$. Write $dd^c\Phi = c(\Omega^+ - \Omega^-)$ with $c \geq 0$ and $\Omega^\pm$ positive closed of mass 1. If S and S' are wedgeable, define the current $S \wedge S'$ by

$$\langle S \wedge S', \Phi \rangle := \langle S', \omega_{FS}^p \wedge \Phi \rangle + c\mathcal{U}_S(S' \wedge \Omega^+) - c\mathcal{U}_S(S' \wedge \Omega^-).$$

A simple computation shows that the definition coincides with the usual wedge-product when S or S' is smooth. One can also prove that the previous definition does not depend on the choice of c, $\Omega^\pm$ and is symmetric with respect to S, S'. If S is of bidegree $(1,1)$, then S, S' are wedgeable if and only if the quasi-potentials of S are integrable with respect to the trace measure of S'. In this case, the above definition coincides with the definition in Appendix A.3. We have the following general result.

Theorem A.48. *Let S_i be positive closed currents of bidegree (p_i, p_i) on $\mathbb{P}^k$ with $1 \leq i \leq m$ and $p_1 + \cdots + p_m \leq k$. Assume that for $1 \leq i \leq m-1$, S_i and $S_{i+1} \wedge \ldots \wedge S_m$ are wedgeable. Then, this condition is symmetric on $S_1, \ldots, S_m$. The wedge-product $S_1 \wedge \ldots \wedge S_m$ is a positive closed current of mass $\|S_1\| \ldots \|S_m\|$ supported on $\mathrm{supp}(S_1) \cap \ldots \cap \mathrm{supp}(S_m)$. It depends linearly on each variable and is symmetric on the variables. If $S_i^{(n)}$ converge to S_i in the Hartogs' sense, then the $S_i^{(n)}$ are wedgeable and $S_1^{(n)} \wedge \ldots \wedge S_m^{(n)}$ converge in the Hartogs' sense to $S_1 \wedge \ldots \wedge S_m$.*

We deduce from this result that wedge-products of PB currents are PB. One can also prove that wedge-products of PC currents are PC. If S_n is defined by analytic sets, they are wedgeable if the intersection of these analytic sets is of codimension $p_1 + \cdots + p_m$. In this case, the intersection in the sense of currents coincides with the intersection of cycles, i.e. is equal to the current of integration on the intersection of cycles where we count the multiplicities. We have the following criterion of wedgeability which contains the case of cycles.

Proposition A.49. *Let S, S' be positive closed currents on $\mathbb{P}^k$ of bidegrees (p,p) and (p', p'). Let W, W' be open sets such that S restricted to W and S' restricted to W' are bounded forms. Assume that $W \cup W'$ is $(p+p')$-concave in the sense that there is a positive closed smooth form of bidegree $(k-p-p'+1, k-p-p'+1)$ with compact support in $W \cup W'$. Then S and S' are wedgeable.*

The following result can be deduced from Theorem A.48.

Corollary A.50. *Let S_i be positive closed $(1,1)$-currents on $\mathbb{P}^k$ with $1 \leq i \leq p$. Assume that for $1 \leq i \leq p-1$, S_i admits a quasi-potential which is integrable with*

respect to the trace measure of $S_{i+1} \wedge \ldots \wedge S_p$. Then, this condition is symmetric on $S_1, \ldots, S_p$. The wedge-product $S_1 \wedge \ldots \wedge S_p$ is a positive closed (p,p)-current of mass $\|S_1\| \ldots \|S_p\|$ supported on $\operatorname{supp}(S_1) \cap \ldots \cap \operatorname{supp}(S_p)$. It depends linearly on each variable and is symmetric on the variables. If $S_i^{(n)}$ converge to S_i in the Hartogs' sense, then the $S_i^{(n)}$ are wedgeable and $S_1^{(n)} \wedge \ldots \wedge S_p^{(n)}$ converge to $S_1 \wedge \ldots \wedge S_p$.

We discuss now currents with Hölder continuous super-potential and moderate currents. The space $\mathscr{C}_{k-p+1}(\mathbb{P}^k)$ admits natural distances $\operatorname{dist}_\alpha$, with $\alpha > 0$, defined by

$$\operatorname{dist}_\alpha(R, R') := \sup_{\|\Phi\|_{\mathscr{C}^\alpha} \leq 1} |\langle R - R', \Phi \rangle|,$$

where Φ is a smooth $(p-1, p-1)$-form on $\mathbb{P}^k$. The norm $\mathscr{C}^\alpha$ on Φ is the sum of the $\mathscr{C}^\alpha$-norms of its coefficients for a fixed atlas of $\mathbb{P}^k$. The topology associated to $\operatorname{dist}_\alpha$ coincides with the weak topology. Using the theory of interpolation between Banach spaces [T1], we obtain for $\beta > \alpha > 0$ that

$$\operatorname{dist}_\beta \leq \operatorname{dist}_\alpha \leq c_{\alpha,\beta} [\operatorname{dist}_\beta]^{\alpha/\beta}$$

where $c_{\alpha,\beta} > 0$ is a constant. So, a function on $\mathscr{C}_{k-p+1}(\mathbb{P}^k)$ is Hölder continuous with respect to $\operatorname{dist}_\alpha$ if and only if it is Hölder continuous with respect to $\operatorname{dist}_\beta$. The following proposition is useful in dynamics.

Proposition A.51. *The wedge-product of positive closed currents on $\mathbb{P}^k$ with Hölder continuous super-potentials has a Hölder continuous super-potential. Let S be a positive closed (p,p)-current with a Hölder continuous super-potential. Then, the Hausdorff dimension of S is strictly larger than $2(k-p)$. Moreover, S is moderate, i.e. for any bounded family $\mathscr{F}$ of d.s.h. functions on $\mathbb{P}^k$, there are constants $c > 0$ and $\alpha > 0$ such that*

$$\int e^{\alpha|u|} d\sigma_S \leq c$$

for every u in $\mathscr{F}$, where σ_S is the trace measure of S.

Exercise A.52. Show that there is a constant $c > 0$ such that

$$\operatorname{cap}(E) \geq \exp(-c/\operatorname{volume}(E)).$$

Hint: use the compactness of $\mathscr{P}_1$ in L^1.

Exercise A.53. Let (u_n) be a sequence of d.s.h. functions such that $\sum \|u_n\|_{\mathrm{DSH}}$ is finite. Show that $\sum u_n$ converge pointwise out of a pluripolar set to a d.s.h. function. Hint: write $u_n = u_n^+ - u_n^-$ with $u_n^\pm \leq 0$, $\|u_n^\pm\|_{\mathrm{DSH}} \lesssim \|u_n\|_{\mathrm{DSH}}$ and $dd^c u_n^\pm \geq -\|u_n\|_{\mathrm{DSH}} \omega_{\mathrm{FS}}$.

Exercise A.54. If χ is a convex increasing function on $\mathbb{R}$ with bounded derivative and u is a d.s.h. function, show that $\chi \circ u$ is d.s.h. If χ is Lipschitz and u is in $W^*(\mathbb{P}^k)$, show that $\chi \circ u$ is in $W^*(\mathbb{P}^k)$. Prove that bounded d.s.h. functions are in $W^*(\mathbb{P}^k)$. Show that $\mathrm{DSH}(\mathbb{P}^k)$ and $W^*(\mathbb{P}^k)$ are stable under the max and min operations.

Exercise A.55. Let   be a non-zero positive measure which is WPC. Define

$$\|u\|^* := |\langle\ ,u\rangle| + \min \|\Theta\|^{1/2}$$

with Θ as above. Show that $\|\cdot\|^*$ defines a norm which is equivalent to $\|\cdot\|_{W^*}$.

Exercise A.56. Show that the capacity of R is positive if and only if R is PB.

Exercise A.57. Let S be a positive closed (p,p)-current of mass 1 with positive Lelong number at a point a. Let H be a hyperplane containing a such that S and $[H]$ are wedgeable. Show that the Lelong number of $S \wedge [H]$ at a is the same if we consider it as a current on $\mathbb{P}^k$ or on H. If R is a positive closed current of bidimension $(p-1,p-1)$ on H, show that $\mathscr{U}_S(R) \leq \mathscr{U}_{S \wedge [H]}(R) + c$ where $c > 0$ is a constant independent of S, R and H. Deduce that PB currents have no positive Lelong numbers.

Exercise A.58. Let K be a compact subset in $\mathbb{C}^k \subset \mathbb{P}^k$. Let $S_1, \ldots, S_p$ be positive closed $(1,1)$-currents on $\mathbb{P}^k$. Assume that their quasi-potentials are bounded on $\mathbb{P}^k \setminus K$. Show that $S_1, \ldots, S_p$ are wedgeable. Show that the wedge-product $S_1 \wedge \ldots \wedge S_p$ is continuous for Hartogs' convergence.

Exercise A.59. Let S and S' be positive closed (p,p)-currents on $\mathbb{P}^k$ such that $S' \leq S$. Assume that S is PB (resp. PC). Show that S' is PB (resp. PC).

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