

Brownian motion on foliated complex surfaces, Brownian motion, and applications

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These lectures are motivated by the dynamical study of differential equations in the complex domain. Most of the topic will concern holomorphic foliations on complex surfaces. In foliation theory, the interplay between geometry and dynamics is what makes the beauty of the subject. In these lectures, we will try to develop this relationship even more.

On the geometrical side, we have generalizations of the foliation cycles introduced by Sullivan: namely the foliated harmonic currents. Those currents permit to think of the foliation as if it were a genuine algebraic curve. For instance, one can associate a homology class, compute intersections with divisors on the surface etc. ... These currents can often be viewed as limits of the (conveniently normalized) currents of integration on large leafwise domains defined via the uniformization of the leaves. This point of view, closely related to Nevanlinna theory, is very fruitful in the applications as we will see.

On the dynamical side, the leafwise Brownian motions (wrt to some hermitian metric on the tangent bundle to the foliation, e.g. coming from uniformization of leaves) generate a Markov process on the complex surface, whose study was begun by Garnett. This Markov process seems to play a determinant role in the dynamics of foliated complex surfaces. One reason is that the Brownian motion in two dimensions is conformally invariant. Another reason is that leafwise Brownian trajectories equidistribute wrt the product of a certain foliated harmonic current times the leafwise volume element. This makes the connection with the geometrical side mentioned above.

One of the main theme that will be developed in these lectures is the construction of numerical invariants that embrace these two aspects (dynamical and geometrical) of foliated complex surfaces. The discussion will emphasize on the definition and properties of the foliated Lyapunov exponent of a harmonic current, which heuristically measures the exponential rate of convergence of leaves toward each other along leafwise Brownian trajectories. A fruitful formula expresses this dynamical invariant in terms of the intersection of some foliated harmonic currents and the normal/canonical bundles of the foliation. This formula is a good illustration of the interplay between geometry and dynamics in foliation theory.